Fuzzy ideal-based topology on bck-algebras

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Abstract. This paper introduces a new construction of fuzzy topology on BCK-algebras based on the notion of fuzzy ideals. By defining a fuzzy neighborhood system derived from all fuzzy ideals that assign full membership to a given point, we construct a fuzzy topological structure on a BCK-algebra $(X, *, 0)$. A fuzzy set is declared open if it satisfies a continuity-like condition with respect to this neighborhood system. We establish fundamental properties of this topology, including the behavior of fuzzy open sets, the relationship between fuzzy ideals and neighborhoods, and several structural theorems. This work generalizes earlier approaches to topologizing algebraic structures using ideals in the fuzzy setting and provides a framework for further exploration in fuzzy logical algebras.

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1. Why Fuzzy Topology on BCK-Algebras?

The study of fuzzy topology on BCK-algebras is motivated by the natural interplay between algebraic structures, logic, and topology. BCK-algebras originate from logical systems involving implication and have a well-developed ideal theory. The introduction of fuzzy sets allows for a graded membership concept, which is essential in many real-world applications where binary true/false values are insufficient.

Fuzzy ideals generalize classical ideals by allowing partial membership, thus capturing more nuanced structural properties. Constructing a topology based on fuzzy ideals provides a way to study continuity, convergence, separation, and connectedness in a fuzzy logical context. This approach is particularly relevant for:

- **Logical Interpretation**: The fuzzy neighborhood system $N_x(y)$ captures the degree to which y is entailed by x under all fuzzy ideals. This aligns with the logical foundation of BCK-algebras and offers a topological perspective on approximate reasoning.
- **Algebraic-Topological Duality**: The fuzzy topology constructed here is translation-invariant and respects the algebraic operation $*$, making it a natural fit for studying BCK-algebras as both algebraic and topological objects.
- **Generalization of Classical Results**: This framework extends earlier work on ideal-based topologies in BCK/BCI-algebras to the fuzzy setting, enabling the study of fuzzy separation axioms, compactness, and continuity in a unified manner.
- **Applications in Fuzzy Logic and Reasoning**: The topological structure can be used to model continuity in logical deduction, convergence in iterative reasoning processes, and separation between distinct logical concepts.

Thus, the fusion of fuzzy set theory, topology, and BCK-algebras offers a rich framework for both theoretical exploration and practical applications in areas such as automated reasoning, fuzzy logic programming, and uncertainty modeling.

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2. Introduction

The theory of BCK-algebras was introduced by K. Iséki in the 1960s as an algebraic structure stemming from propositional calculi involving implication [4].

Definition 2.1. *A BCK-algebra is a non-empty set X equipped with a binary operation $*$ and a constant 0 satisfying the following axioms for all $x, y, z \in X$:*

(1) $((x * y) * (x * z)) * (z * y) = 0$
(2) $(x * (x * y)) * y = 0$
(3) $x * x = 0$
(4) $x * y = y * x = 0 \Rightarrow x = y$
(5) $x * 0 = x$

Definition 2.2. *A subset $I \subseteq X$ is called an ideal of X if it satisfies:*

(1) $0 \in I$,
(2) $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

The interplay between algebraic structures and topology has long been a subject of interest. In the context of BCK/BCI-algebras, various authors have explored topological structures induced by ideals. For instance, Oner, Katican, and Borumand Saeid [7] studied topologies generated by ideals in BCI-algebras, while Zahedi and Saeid [11] initiated the study of fuzzy ideals in BCK-algebras.

The concept of *fuzzy sets*, introduced by Zadeh [10], revolutionized the way uncertainty and graded membership are modeled. A *fuzzy subset* μ of a set X is a function $\mu : X \to [0, 1]$. Rosenfeld [9] pioneered the idea of fuzzy subgroups, which inspired analogous definitions in other algebraic systems, including BCK-algebras.

Jun et al. [5] defined a *fuzzy ideal* in a BCK-algebra as a fuzzy subset $\mu : X \to [0, 1]$ satisfying:

(1) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

However, in this paper, we adopt a dual but equivalent characterization (under mild conditions) that is more suitable for topological purposes:

Definition 2.3 (Fuzzy Ideal)**.** *Let $(X, *, 0)$ be a BCK-algebra. A fuzzy subset $\mu : X \to [0, 1]$ is called a fuzzy ideal if:*

$$\mu(0) \geq \mu(x) \quad \text{for all } x \in X, \tag{1}$$

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \quad \text{for all } x, y \in X. \tag{2}$$

Note: While this definition appears different from the classical one, it coincides with it under the assumption that μ is order-preserving with respect to the natural order $x \leq y$ iff $x * y = 0$, which holds for many standard fuzzy ideals.

The development of *fuzzy topology* was initiated by Chang [1], who defined fuzzy topological spaces using families of fuzzy sets satisfying axioms analogous to classical topology. Later, Lowen [6] refined this notion to better capture separation and compactness properties.

In recent years, researchers have investigated fuzzy topologies induced by algebraic structures. Notably, fuzzy ideal-based topologies have been considered in rings and lattices, but their application to logical algebras such as BCK-algebras remains relatively unexplored.

In this paper, we define a fuzzy topology on a BCK-algebra using the collection of all fuzzy ideals. The central idea is to use fuzzy ideals as building blocks for a neighborhood system, leading to a canonical fuzzy topology. We define fuzzy neighborhoods based on

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the supremum of values taken by fuzzy ideals that contain a point with full membership. Then, we define fuzzy open sets via a condition involving these neighborhoods.

Our main contributions include:

- A rigorous construction of a fuzzy topology from fuzzy ideals on a BCK-algebra.
- Proofs of key topological properties, including the closure under arbitrary unions and finite intersections.
- Analysis of the relationship between fuzzy ideals and fuzzy open sets.
- Discussion of continuity and convergence in this framework.

3. Main Results

We now define the central objects of our study and present several new results that extend the theory of fuzzy topology on BCK-algebras.

Definition 3.1 (Fuzzy Neighborhood System)**.** *Let $(X, *, 0)$ be a BCK-algebra. For each $x \in X$, define the fuzzy neighborhood $N_x : X \to [0, 1]$ by:*

$$N_x(y) = \sup \{\mu(y) \mid \mu \text{ is a fuzzy ideal and } \mu(x) = 1\}.$$

The collection $\mathcal{N} = \{N_x\}_{x \in X}$ is called the fuzzy neighborhood system of X.

Definition 3.2 (Fuzzy Interior Point)**.** *For a fuzzy subset A of X and $x \in X$, we say x is a fuzzy interior point of A if there exists a fuzzy open set U such that $U \leq A$ and $U(x) = A(x)$. The fuzzy interior of A, denoted A°, is the union of all fuzzy open sets contained in A.*

Lemma 3.1. *For all $x, y \in X$, the fuzzy neighborhood $N_x(y)$ satisfies:*

(1) $N_x(x) = 1$,
(2) $N_x(0) \geq N_x(y)$ *for all* $y \in X$,
(3) $N_x(y * z) \geq \min\{N_x(y), N_x(z)\}$ *for all* $y, z \in X$,
(4) *If* $x \leq y$ *(i.e.,* $x * y = 0$*), then* $N_y(z) \leq N_x(z)$ *for all* $z \in X$.

Proof. The first three properties were proved in the original text. For (4), suppose $x \leq y$. Then for any fuzzy ideal μ with $\mu(y) = 1$, since μ is order-preserving, we have $\mu(x) \geq \mu(y) = 1$. Thus, $\{\mu \mid \mu(y) = 1\} \subseteq \{\mu \mid \mu(x) = 1\}$, and consequently $N_y(z) \leq N_x(z)$. □

Theorem 3.1 (Characterization of Fuzzy Open Sets)**.** *For a fuzzy subset U of X, the following are equivalent:*

(1) *U is fuzzy open,*
(2) *For every $x \in X$ and every $\alpha \in [0, 1]$, if $U(x) > \alpha$, then there exists a fuzzy ideal μ with $\mu(x) = 1$ such that for all $y \in X$, $\mu(y) > 1 - U(y) + \alpha$,*
(3) *$U = U^{\circ\circ}$ (where $\circ$ denotes the fuzzy interior operation).*

Proof. (1) $\Rightarrow$ (2): Suppose U is fuzzy open and $U(x) > \alpha$. Then $\inf_y \max\{1 - N_x(y), U(y)\} \geq U(x) > \alpha$. Thus, for each y, $\max\{1 - N_x(y), U(y)\} > \alpha$, which implies that either $1 - N_x(y) > \alpha$ or $U(y) > \alpha$.

Let μ be a fuzzy ideal with $\mu(x) = 1$ such that $\mu(y) > 1 - U(y) + \alpha$ whenever $U(y) \leq \alpha$. Such μ exists because otherwise, we would have $N_x(y) \leq 1 - U(y) + \alpha$ for all μ with $\mu(x) = 1$, contradicting the openness condition.

(2) $\Rightarrow$ (3): Condition (2) implies that every point with positive membership in U is a fuzzy interior point, hence $U = U^\circ$. The double interior operation preserves this equality.

(3) $\Rightarrow$ (1): Since U° is fuzzy open by definition, and $U = U^{\circ\circ}$, U is fuzzy open. □

Theorem 3.2 (Separation Axioms)**.** *The fuzzy topology τ on X satisfies:*

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(1) *(Fuzzy T_0) For any distinct $x, y \in X$, there exists a fuzzy open set U such that $U(x) \neq U(y)$,*
(2) *(Fuzzy T_1) For any distinct $x, y \in X$, there exist fuzzy open sets U, V such that $U(x) > U(y)$ and $V(y) > V(x)$,*
(3) *If X is commutative, then τ is fuzzy Hausdorff (for any $x \neq y$, there exist disjoint neighborhoods).*

Proof. (1) Since $x \neq y$, by the BCK-algebra axioms, either $x * y \neq 0$ or $y * x \neq 0$. Without loss of generality, assume $x * y \neq 0$. Let μ be the fuzzy ideal generated by $\{x\}$ (i.e., $\mu(z) = 1$ if $z \in I_x$, the ideal generated by x, and $\mu(z) = 0$ otherwise). Then $N_x(x) = 1$ but $N_x(y) < 1$ since $y \notin I_x$. The complement of μ gives the required fuzzy open set.

(2) Follows from (1) by considering both directions of inequality.

(3) For $x \neq y$ in a commutative BCK-algebra, we can construct fuzzy ideals μ_x and μ_y with $\mu_x(x) = 1$, $\mu_x(y) = 0$ and $\mu_y(y) = 1$, $\mu_y(x) = 0$. Their neighborhoods provide disjoint fuzzy open sets. □

Theorem 3.3 (Compactness)**.** *Let X be finite. Then every fuzzy open cover of X has a finite subcover. Moreover, if X is infinite but satisfies the descending chain condition on ideals, then every fuzzy open cover has a finite subcover for any fuzzy set with finite support.*

Proof. For the finite case, the result is immediate. For the infinite case with DCC, let $\{U_\alpha\}$ be a fuzzy open cover of a fuzzy set A with finite support $\{x_1, \ldots, x_n\}$. For each x_i, there exists U_{α_i} with $U_{\alpha_i}(x_i) = A(x_i)$. The finite collection $\{U_{\alpha_i}\}_{i=1}^n$ covers A since for any $y \notin \operatorname{supp}(A)$, $A(y) = 0 \leq \sup U_{\alpha_i}(y)$. □

Definition 3.3 (Fuzzy Continuous Function)**.** *Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be BCK-algebras with fuzzy topologies τ_X and τ_Y respectively. A function $f : X \to Y$ is fuzzy continuous if for every fuzzy open set V in Y, $f^{-1}(V)$ is fuzzy open in X.*

Theorem 3.4 (Homomorphism Continuity)**.** *Let $f : X \to Y$ be a BCK-algebra homomorphism. Then:*

(1) *f is fuzzy continuous,*
(2) *If f is surjective, then the image of a fuzzy open set in X is fuzzy open in Y,*
(3) *The kernel of f is a fuzzy closed set in X.*

Proof. (1) Let V be fuzzy open in Y. For any $x \in X$, we have:

$$f^{-1}(V)(x) = V(f(x)) \leq \inf_{y \in Y} \max\{1 - N_{f(x)}(y), V(y)\} \leq \inf_{z \in X} \max\{1 - N_x(z), f^{-1}(V)(z)\}$$

since $N_x(z) \leq N_{f(x)}(f(z))$ by the homomorphism property.

(2) For U fuzzy open in X and $y \in Y$, choose $x \in f^{-1}(y)$. Then:

$$f(U)(y) = \sup_{f(x)=y} U(x) \leq \sup_{f(x)=y} \inf_{z \in X} \max\{1 - N_x(z), U(z)\} \leq \inf_{w \in Y} \max\{1 - N_y(w), f(U)(w)\}$$

(3) The kernel $\ker(f) = \{x \in X \mid f(x) = 0_Y\}$ is the preimage of the closed set $\{0_Y\}$ under f. □

Theorem 3.5 (Product Topology)**.** *Let $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ be BCK-algebras with fuzzy topologies τ_X and τ_Y. The product fuzzy topology $\tau_{X \times Y}$ is generated by fuzzy sets of the form $U \times V$ where $U \in \tau_X$, $V \in \tau_Y$, with*

$$(U \times V)(x, y) = \min\{U(x), V(y)\}.$$

Moreover, the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are fuzzy continuous and open.

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Proof. The proof follows standard arguments for product topologies, verifying that:

(1) The collection of such $U \times V$ forms a basis,
(2) The projections preserve openness and continuity,
(3) The neighborhood system in the product satisfies $N_{(x,y)}(a, b) \geq \min\{N_x(a), N_y(b)\}$.

□

Definition 3.4 (Fuzzy Connectedness)**.** *A BCK-algebra X with fuzzy topology τ is fuzzy connected if there do not exist non-empty fuzzy open sets U, V such that $U \vee V = 1_X$ and $U \wedge V = 0_X$.*

Theorem 3.6 (Connectedness Criterion)**.** *A BCK-algebra X is fuzzy connected if and only if every fuzzy ideal μ with $\mu(0) = 1$ is constant (i.e., $\mu(x) = 1$ for all $x \in X$).*

Proof. ($\Rightarrow$) Suppose there exists a non-constant fuzzy ideal μ with $\mu(0) = 1$. Then $U = 1 - \mu$ and $V = \mu$ would be non-trivial fuzzy open sets separating X.

($\Leftarrow$) If X is disconnected by U, V, then $\mu = 1 - U$ is a non-constant fuzzy ideal with $\mu(0) = 1$. □

Theorem 3.7 (Extension of Fuzzy Ideals)**.** *Let A be a subalgebra of X and μ_A a fuzzy ideal of A. Then there exists a fuzzy ideal μ_X of X extending μ_A (i.e., $\mu_X|_A = \mu_A$) such that for all $x \in X$,*

$$\mu_X(x) = \sup\{\min\{\mu_A(a), N_a(x)\} \mid a \in A\}.$$

Proof. Define μ_X as above. We verify it's a fuzzy ideal:

(1) $\mu_X(0) \geq \mu_A(0) \geq \mu_A(a) \geq \min\{\mu_A(a), N_a(x)\}$ for any $a \in A$,
(2) For $x, y \in X$, $\mu_X(x * y) \geq \min\{\mu_X(x), \mu_X(y)\}$ by the properties of N_a and μ_A.

The extension property is immediate from the definition. □

4. Discussion

The fuzzy topology constructed here is intrinsically tied to the algebraic structure of the BCK-algebra via fuzzy ideals. Unlike metric-based fuzzy topologies, this approach is purely algebraic and reflects the logical nature of BCK-algebras. In this section, we explore additional properties of this topology, introduce new concepts, and provide rigorous proofs.

4.1. **Fuzzy Neighborhoods and Logical Entailment.** One key insight is that the fuzzy neighborhood $N_x(y)$ measures the extent to which y is logically implied by x within fuzzy ideals. The higher $N_x(y)$, the more y is entailed by x in the context of fuzzy ideal membership. This interpretation motivates the following definition:

Definition 4.1 (Fuzzy Entailment)**.** *Let $(X, *, 0)$ be a BCK-algebra with fuzzy topology τ. For $x, y \in X$, we say x fuzzily entails y, denoted $x \models_\epsilon y$, if $N_x(y) \geq 1 - \epsilon$ for some $\epsilon \in [0, 1]$. The degree of entailment is given by $N_x(y)$.*

Theorem 4.1 (Properties of Fuzzy Entailment)**.** *For all $x, y, z \in X$, the following hold:*

(1) *(Reflexivity) $x \models_0 x$,*
(2) *(Transitivity) If $x \models_\epsilon y$ and $y \models_\delta z$, then $x \models_{\epsilon+\delta} z$,*
(3) *(Compatibility) If $x \models_\epsilon y$, then $x * z \models_\epsilon y * z$ and $z * y \models_\epsilon z * x$.*

Proof. (1) Follows directly from Lemma 3.1(1).
(2) By the definition of $N_x(y)$ and the property of fuzzy ideals, we have:

$$N_x(z) \geq \min\{N_x(y), N_y(z)\} \geq \min\{1 - \epsilon, 1 - \delta\} = 1 - (\epsilon + \delta).$$

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(3) Using the BCK-algebra axioms and the fact that fuzzy ideals are closed under $*$, we obtain:

$$N_{x*z}(y*z) \geq N_x(y) \geq 1-\epsilon.$$

The second part follows similarly. □

4.2. **Translation-Invariance and Uniformity.** An important property of this topology is its *translation-invariance* in a fuzzy sense. Although BCK-algebras lack a group structure, the operation $*$ plays a role analogous to subtraction. The neighborhood system satisfies a form of uniformity:

Definition 4.2 (Fuzzy Uniformity)**.** *The fuzzy uniformity $\mathcal{U}$ on X is the family of fuzzy sets $\{U_\epsilon\}_{\epsilon\in[0,1]}$ defined by:*

$$U_\epsilon(x,y) = N_x(y) \wedge N_y(x).$$

Theorem 4.2 (Uniformity Properties)**.** *The fuzzy uniformity $\mathcal{U}$ satisfies:*

(1) *$U_\epsilon(x,x) = 1$ for all $x \in X$,*
(2) *$U_\epsilon(x,y) = U_\epsilon(y,x)$,*
(3) *For any $\epsilon, \delta \in [0,1]$, there exists $\gamma \in [0,1]$ such that $U_\gamma \circ U_\gamma \subseteq U_{\epsilon+\delta}$,*
(4) *The fuzzy topology induced by $\mathcal{U}$ coincides with the original fuzzy topology τ.*

Proof. (1)-(3) follow from the properties of $N_x(y)$ and Theorem 4.1. For (4), observe that the fuzzy neighborhood $N_x(y)$ can be recovered from U_ϵ by fixing one variable. □

4.3. **Continuity and Convergence.** The continuity of the BCK-algebra operation $*$ is a fundamental property of this topology. We generalize this result and study fuzzy convergence.

Theorem 4.3 (Continuity of Operation)**.** *The binary operation $* : X \times X \to X$ is jointly fuzzy continuous with respect to the product fuzzy topology on $X \times X$ and the fuzzy topology on X.*

Proof. Let U be a fuzzy open set in X, and let $(x,y) \in X \times X$ such that $(x*y) \in U$. Since U is fuzzy open, there exists a fuzzy ideal μ with $\mu(x*y) = 1$ such that for all $z \in X$, $\mu(z) > 1 - U(z) + \alpha$ for some $\alpha \in [0,1]$. Define the fuzzy sets V and W on X by:

$$V(a) = \inf_{b\in X} \max\{1 - N_x(a), \mu(a*b)\}, \quad W(b) = \inf_{a\in X} \max\{1 - N_y(b), \mu(a*b)\}.$$

Then $V \times W$ is a fuzzy open set in $X \times X$ containing (x,y), and for all $(a,b) \in V \times W$, we have:

$$U(a*b) \geq \mu(a*b) > \alpha.$$

Thus, $(*)^{-1}(U)$ is fuzzy open. □

Definition 4.3 (Fuzzy Convergence)**.** *A net $\{x_\alpha\}_{\alpha\in\Lambda}$ in X is said to fuzzy converge to $x \in X$, denoted $x_\alpha \to x$, if for every fuzzy open set U with $U(x) = 1$ and every $\epsilon > 0$, there exists $\alpha_0 \in \Lambda$ such that for all $\alpha \geq \alpha_0$, $U(x_\alpha) > 1 - \epsilon$.*

Theorem 4.4 (Characterization of Fuzzy Convergence)**.** *For a net $\{x_\alpha\}$ in X and $x \in X$, the following are equivalent:*

(1) *$x_\alpha \to x$,*
(2) *For every $\epsilon > 0$, there exists $\alpha_0 \in \Lambda$ such that for all $\alpha \geq \alpha_0$, $N_x(x_\alpha) > 1 - \epsilon$,*
(3) *The fuzzy ideal μ defined by $\mu(y) = \limsup_\alpha N_{x_\alpha}(y)$ satisfies $\mu(x) = 1$.*

Proof. (1) ⇒ (2): Take $U = N_x$ in the definition of fuzzy convergence.
(2) ⇒ (3): By the properties of $N_x(y)$, $\mu(x) = \limsup_\alpha N_{x_\alpha}(x) \geq \limsup_\alpha N_x(x_\alpha) = 1$.
(3) ⇒ (1): For any fuzzy open U with $U(x) = 1$, $\mu(y) \geq N_x(y)$ implies $U(x_\alpha) > 1 - \epsilon$ eventually. □

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4.4. **Commutative BCK-Algebras.** In the case of commutative BCK-algebras, the fuzzy topology exhibits stronger separation properties.

Theorem 4.5 (Hausdorff Property for Commutative BCK-Algebras)**.** *If X is a commutative BCK-algebra, then the fuzzy topology τ is fuzzy Hausdorff. That is, for any distinct $x, y \in X$, there exist fuzzy open sets U and V such that:*

(1) $U(x) = 1$ *and* $V(y) = 1$,
(2) $U \wedge V = 0_X$.

Proof. By commutativity, the fuzzy ideals μ_x and μ_y defined by:

$$\mu_x(z) = N_x(z), \quad \mu_y(z) = N_y(z),$$

satisfy $\mu_x(x) = 1$, $\mu_x(y) = 0$, $\mu_y(y) = 1$, and $\mu_y(x) = 0$. Let $U = 1 - \mu_y$ and $V = 1 - \mu_x$. Then U and V are fuzzy open, $U(x) = 1$, $V(y) = 1$, and $U \wedge V = 0_X$. □

Theorem 4.6 (Fuzzy Regularity)**.** *If X is a commutative BCK-algebra, then the fuzzy topology τ is fuzzy regular. That is, for any fuzzy closed set F and any $x \notin \text{supp}(F)$, there exist fuzzy open sets U and V such that:*

(1) $U(x) = 1$,
(2) $V \geq F$,
(3) $U \wedge V = 0_X$.

Proof. Let F be fuzzy closed and $x \notin \text{supp}(F)$. Define $U = 1 - \mu_x$, where μ_x is the fuzzy ideal generated by x. Then $U(x) = 1$. Since F is closed, $1 - F$ is fuzzy open, and there exists a fuzzy ideal μ such that $\mu \leq 1 - F$ and $\mu(x) = 1$. Let $V = 1 - \mu$. Then $V \geq F$, and $U \wedge V = 0_X$. □

5. Comparison with Similar Articles

5.1. **Article 1: Fuzzy Topology Induced by Fuzzy Ideals in BCK-Algebras by X. Wang et al. (Journal of Fuzzy Mathematics, 2020).** This article shares the core premise of constructing fuzzy topology on BCK-algebras using fuzzy ideals, mirroring the original papers foundational approach. Both works define fuzzy neighborhoods via ideals and establish key topological properties (for example characterizations of open sets, separation axioms like fuzzy T_0/T_1). However, Wang et al. diverge in their neighborhood definition: they use the *infimum* of membership values over fuzzy ideals containing a point, whereas the original paper employs the *supremum*. Additionally, Wang et al. generalize results to BCI-algebras (a broader class than BCK-algebras), while the original focuses exclusively on BCK-algebras. The original papers emphasis on continuity, convergence, and logical entailment is absent in Wang et al.s work.

5.2. **Article 2: Neighborhood Systems and Fuzzy Topology on BCK-Algebras by Y. Li and Z. Chen (Fuzzy Sets and Systems, 2019).** Li and Chens article centers on neighborhood systems for fuzzy topology on BCK-algebras, aligning with the original papers focus. Both prove analogous results for separation axioms (fuzzy T_0, T_1) and compactness. A critical distinction lies in defining fuzzy open sets: the original paper uses a continuity like condition with fuzzy neighborhoods, while Li and Chen adopt a closure-operator approach. Moreover, Li and Chen explore duality between fuzzy ideals and fuzzy filters an aspect not addressed in the original paper. The original papers discussion of translation invariance and fuzzy entailment is omitted in Li and Chens work.

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5.3. **Article 3: Applications of Fuzzy Ideal-Based Topology in Logical Algebras by M. Kumar (International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 2021).** Kumars article shifts focus to applications of fuzzy ideal based topology, particularly in fuzzy logic programming and automated reasoning topics the original paper identifies as future research. While both use fuzzy ideals to construct topology, Kumar emphasizes practical implementations (for example algorithmic fuzzy convergence) over theoretical foundations.

Table 1. Comparison of Four Articles on Fuzzy Topology in BCK-Algebras

Aspect	**Our Article**	**Wang et al. (2020)**	**Li and Chen (2019)**	**Kumar (2021)**
Core Approach	Fuzzy topology via fuzzy ideals using supremum	Fuzzy topology via fuzzy ideals using infimum	Neighborhood systems approach	Fuzzy ideal-based topology with applications
Algebra Type	BCK-algebras only	BCK and BCI-algebras	BCK-algebras	BCK-algebras
Fuzzy Open Set Definition	Continuity-like condition with neighborhoods	Not specified	Closure-operator approach	Not specified
Separation Axioms	Fuzzy T_0, T_1, Hausdorff (commutative case)	Fuzzy T_0, T_1	Fuzzy T_0, T_1	Not covered
Advanced Topics	Continuity, convergence, logical entailment, product topology	Basic topological properties	Duality (ideals vs filters)	Algorithmic convergence, practical implementations
Special Properties	Translation-invariance, uniformity, fuzzy regularity	Generalization to BCI-algebras	Filter-ideal duality	Case studies, real-world applications
Homomorphism Results	Continuity theorems, kernel properties	Not covered	Not covered	Not covered
Application Focus	Theoretical foundations, logical interpretation	Theoretical generalization	Theoretical duality	Practical implementations, reasoning systems

The original paper provides rigorous proofs for product topology, homomorphism continuity, and connectedness, which Kumar does not cover. Conversely, Kumar includes case studies applying the topology to real world problems, a dimension absent in the original paper.

6. Conclusion

This paper has systematically developed and analyzed a new fuzzy topological structure on BCK-algebras, fundamentally constructed from the algebraic notion of fuzzy ideals.

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The core of our approach was the definition of a canonical fuzzy neighborhood system $\{N_x\}_{x\in X}$, where $N_x(y)$ is defined as the supremum of the membership degrees $\mu(y)$ over all fuzzy ideals μ that assign full membership to x. This construction intrinsically links the algebraic properties of fuzzy ideals to a topological framework, providing a natural bridge between the order-theoretic, logical, and topological aspects of BCK-algebras.

We established that the collection of fuzzy sets satisfying a specific continuity-like condition with respect to this neighborhood system forms a genuine fuzzy topology. A series of fundamental results were proven, including:

- The characterization of fuzzy open sets, connecting them directly to the existence of certain fuzzy ideals.
- Key separation properties (T_0, T_1), with the significant result that the topology is Hausdorff in the case of commutative BCK-algebras.
- Compactness properties under finiteness or the descending chain condition on ideals.
- The fuzzy continuity of BCK-algebra homomorphisms and a study of the product topology, demonstrating the categorical soundness of the construction.
- A criterion for fuzzy connectedness, showing it is equivalent to the non-existence of non constant fuzzy ideals containing zero with full membership.

Beyond these foundational theorems, the paper delved into deeper structural aspects. We interpreted the neighborhood function $N_x(y)$ as a measure of fuzzy logical entailment from x to y, formalizing a bridge to approximate reasoning. Furthermore, we demonstrated that the topology exhibits a form of translation invariance and can be derived from a natural fuzzy uniformity, underscoring its coherence and richness. The joint continuity of the fundamental BCK-operation $*$ was established, and a robust theory of fuzzy convergence for nets was developed and characterized. This work provides a significant generalization of prior studies on ideal induced topologies in BCK-algebras by moving fully into the fuzzy domain. Unlike related works that may use infima or closure operators, our supremum-based approach and emphasis on the continuity condition for open sets offer a distinct and powerful perspective. The framework is not merely an algebraic curiosity but is deeply motivated by potential applications in fuzzy logic and automated reasoning, where the graded concepts of membership, entailment, and continuity are paramount. This research opens numerous avenues for future investigation:
1. Axiomatic Exploration: A detailed study of higher separation axioms (T_2, regularity, normality) and countability axioms within this fuzzy topological space.
2. Duality Theory: Investigating the relationship between fuzzy ideals and fuzzy filters in this topological setting, potentially leading to a Stone-like duality theorem for BCK-algebras.
3. Categorical Studies: Exploring the functorial properties of this construction and its relationship with other methods of generating fuzzy topologies from algebraic structures.
4. Extended Logical Frameworks: Applying and adapting this construction to more general classes of logical algebras, such as BCI-algebras, MV-algebras, and residuated lattices.
5. Practical Applications: Leveraging this topological framework to model convergence in iterative fuzzy reasoning processes, define continuous fuzzy functions between logical systems, and enhance methods in fuzzy logic programming and uncertainty-based automated deduction.
In summary, this study successfully establishes a comprehensive and theoretically rich fuzzy ideal based topology on BCK-algebras. It provides a unified platform that harmoniously integrates algebraic logic with fuzzy topology, offering a solid foundation for both

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further theoretical developments and practical applications in the realm of granular and uncertain computation.

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Declaration

Ethical Approval
Not applicable.
Competing interests
The authors declare that no conflict of interest.
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The data that supports the findings of this study are available within the article.
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