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ON THE CARTAN TORSION OF QUBE (α, β)-METRICS

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Abstract.

The theory of m-th root Finsler metrics has been applied to Ecology, Biology, Seismic Ray Theory, Gravitation, etc. It is regarded as a direct generalization of Riemannian metric in a sense, that is, the second root metric is a Riemannian metric. On the other hand, the Riemannian curvature faithfully reveals the local geometric properties of a Riemann-Finsler metric. In this paper, we will study the class of quintic (α, β)-metrics. We show that 3-th root (α, β)-metrics has a unbound Cartan torsion. Also, we focus on the class of 3-th root (α, β)-metrics. We will study the bound Cartan torsion for a 3-th, 4-th and 5-th root (α, β)–metrics.

1. Introduction

A deep consideration of the works which has done shows that all the results obtained for the bounded Cartan torsion concepts are about a specific class of Finsler metrics, namely, the class of (α, β)–metrics. An (α, β)–metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M [1–3].

One can study the class of m–th root Finsler metrics in order to find bound Cartan torsion. The theory of m–th root Finsler metric has been developed by Shimada and Matsumoto which is applied to biology as an ecological metric [4, 5]. Let M be an n-dimensional C^∞ manifold, TM its tangent bundle. If $F = \sqrt[m]{A}$ be a Finsler metric on M, where $A = A(x, y)$ is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices. Then, F is called an m–th root Finsler metric (see [3, 5? ? ? ? ? –9]). Recall that Berwald-Moór metric is the special m–th root Finsler metric with $F = \sqrt[m]{y^1 y^2 \dots y^m}$ [4, 10, 11]. Physical studies due to Asanov, Pavlov and their co-workers showed the important role played by the Berwald-Moór metric in the theory of space-time structure and gravitation as well as in unified gauge field theories [12–14]. In [15], Balan proved that the Berwald-Moór structures are pseudo-Finsler of Lorentz type and for co-isotropic submanifolds of Berwald-Moór spaces presented the Gauss-Weingarten, Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations.

Among the class of m-th root Finsler metrics, 4-th root metrics have an important and special role in Finsler geometry. In [16], Tayebi found the necessary and sufficient condition under which a generalized 4-th root metric is of isotropic scalar curvature. He obtained the necessary and sufficient condition under which the conformal change of a generalized 4-th root metric is of isotropic scalar curvature. Also, he showed that every 4-th root metric of weakly isotropic flag curvature has vanishing scalar curvature. Tayebi gave the necessary and sufficient condition under which the conformal change of a 4-th root metric is of isotropic scalar curvature [17]. In [18], Tayebi considered an exponential change of 4-th root Finsler metrics $\bar{F} = e^{\beta/F} F$ and found necessary and sufficient condition under which $\bar{F}$ be locally projectively flat [18]. Also, he obtained

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necessary and sufficient conditions under which $\bar{F}$ be locally dually flat. In [19], Tayebi-Amini-Najafi found the necessary and sufficient condition under which a quartic (α,β)-metric $F=\sqrt[4]{c_1\alpha^4+c_2\alpha^2\beta^2+c_3\beta^4}$ is conformally Berwald, where c_i $(1\leq i\leq 3)$ are real constants. They obtained the necessary and sufficient condition under which a cubic (α,β)–metric $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ is conformally Berwald.

In this paper, we focus on the class of 3-th root (α,β)-metrics. We will study the bound Cartan torsion for a 3-th, 4-th and 5-th root (α,β)–metrics. Essentially we will prove:

Theorem 1.1. *Let $F=\sqrt[3]{c_1\alpha^2\beta+c_2\beta^3}$ be a 3-th root (α,β)-metric. Then F has unbounded Cartan torsion.*

2. Preliminaries

For a Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on $TM_0:=TM-\{0\}$, which in a standard coordinates (x^i,y^i) for TM_0 is given by

$$\mathbf{G}=y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i:=\frac{1}{4}g^{il}\Big[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k-\frac{\partial F^2}{\partial x^l}\Big],\qquad y\in T_xM. \tag{2.1}$$

$\mathbf{G}$ is called the spray associated to (M,F).

The function $F=\alpha\phi(s)$ is a Finsler metric for any $\alpha=\sqrt{a_{ij}y^iy^j}$ and any $\beta=b_iy^i$ with $||\beta_x||_\alpha<b_0$ if and only if ϕ is a positive C^∞ function on $(-b_0,b_0)$ satisfying the following condition:

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\qquad |s|\leq b<b_0. \tag{2.2}$$

From (2.2), one can see that $\phi=\phi(s)$ must satisfy

$$\phi(s)-s\phi'(s)>0,\qquad |s|<b_0.$$

For more details, see [20]. A Finsler metric F on a manifold M is called an (α,β)–metric if it is expressed as $F=\alpha\phi(s)$ with $||\beta_x||_\alpha<b_0$, where $\phi(s)$ is a positive C^∞ on $(-b_0,b_0)$ satisfying (2.2).

3. Proof of Theorem 1.1

Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)–metric, where $\phi=\phi(s)$ is a C^∞ on $(-b_0,b_0)$ with certain regularity, $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1–form on a manifold M. Let $G^i=G^i(x,y)$ and $G^i_\alpha=G^i_\alpha(x,y)$ denote the coefficients of F and α, respectively, in the same coordinate system. For an (α,β)–metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j:=db_i-b_j\theta_i^j$, where $\theta^i:=dx^i$ and $\theta_i^j:=\Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$\begin{aligned}&r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),\\&r^i{}_j:=a^{is}r_{sj},\quad s^i{}_j:=a^{is}s_{sj},\quad q_{ij}:=r_{is}s^s{}_j,\quad t_{ij}:=t_{ik}s^k{}_j,\\&r_j:=b^ir_{ij},\quad s_j:=b^is_{ij},\quad s^i=a^{ik}s_k,\quad q_j:=b^iq_{ij},\quad t_j:=b^it_{ij},\end{aligned}$$

where "|" denotes the covariant derivative with respect to the Levi-Civita connection of α and $b^i:=a^{ij}b_j$, a^{ij} is the inverse of a_{ij}. We define $r_{i0}:=r_{ij}y^j$ and $r_{00}:=r_{ij}y^iy^j$, etc [21]. In order to prove Theorem 1.1, first we show the following.

Lemma 3.1. ([22])*Let* $F = \sqrt[3]{a_{ijk}y^iy^jy^k}$ *be a cubic Finsler metric on a manifold* M *of dimension dim* $(M) \geq$. *If* F *is a function of a non-degenerate quadratic form* $\alpha = \sqrt{a_{ij}y^iy^j}$ *and a* 1*-form* $\beta = b_i(x)y^i$ *which is homogeneous in* α *and* β *of degree one, then it is written in the following form*

$$F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$$

where c_1 *and* c_2 *are real constants.*

In [23], Tayebi and Sadeghi proved that the norm of Cartan and mean Cartan torsion of (α, β)-metric on manifold M of dimension $n \geq 3$ satisfy following relation

$$\parallel \mathbf{C} \parallel = \sqrt{\frac{3p^2 + 6pq + (n+1)q^2}{n+1}} \parallel \mathbf{I} \parallel$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on TM satisfying $p + q = 1$ and assert by

$$\begin{aligned}
p :=& \frac{n+1}{aA}\{s(\phi\phi'' + \phi'\phi') - \phi\phi'\},\\
a :=& \phi(\phi - s\phi'),\\
A :=& (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{-3s\phi'' + (b^2 - s^2)\phi'''}{\phi - s\phi' + (b^2 - s^2)\phi''}.
\end{aligned}$$

A formula for the mean Cartan torsion of (α, β)-metrics obtained as follows

$$\mathbf{I_i} = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i).$$

For 5-th root (α, β)-metric we calculated the I_i, $g_{ij}I_iI_jF^2$ and $\sqrt{3p^2 + 6pq + (n+1)q^2}$ respectively

$$\begin{aligned}
A :=& c_1s^7(9c_1(s(c_1 + c_2s^2))^{\frac{2}{3}} + 9(s(c_1 + c_2s^2))^{\frac{2}{3}}c_2s^2 + 2b^2c_1^2 + 6b^2c_1c_2s^2 - 2s^2c_1^2 - 6s^4c_1c_2)^2,\\
& - 2nc_1^2b2 + 6b2c_1^2)^2,\\
B :=& 6s^2c_1^2 + 6s^4c_1c_2 - b^2c_1^2 + 12b^2c_1c_2s^2 + 9b^2c_2^2s^4 + 4b^2sc_1^2\\
E :=& (27s^6c_2^2 + 27c_2^2s^5 + 18s^4c_1c_2 + 18c_1c_2s^3 + 6b2c_1c_2s^2 + 6s^2nc_1b2c_2 + 8s^2nc_1^2 - 9s^2c_1^2 - 9c_1^2s,
\end{aligned}$$

$$Ii := \frac{c_1(4nc_1s^2 + 3nb2c_2s^2 - nb2c_1 + 3b2c_1 + 3b2c_2s^2)}{3(s(c_1 + c_2s^2)(4c_1s^2 + 3b2c_2s^2 - b2c_1)\alpha^2)}$$

$$\begin{aligned}
g_{ij}I_iI_jF^2 =& \frac{3}{32A}\Big\{(b^2 - s^2)(18s^2nc_1(s(c_1 + c_2s^2))^{\frac{2}{3}} + 18s^4n(s(c_1 + c_2s^2))^{\frac{2}{3}}c_2 + 4s^2nb^2c_1^2\\
& + 12s^4nb^2c_1c_2 - 4s^4nc_1^2 - 12s^6nc_1c_2 + 9\sqrt[3]{(s(c_1 + c_2s^2))^2}b^2c_1\\
& + 9(\sqrt[3]{(s(c_1 + c_2s^2))^2}b^2c_2s^2)^2\sqrt[3]{s(c_1 + c_2s^2)})((c_1s + c_2s^3)^{\frac{1}{3}} - \frac{s(c_1 + 3c_2s^2)}{3(c_1s + c_2s^3)^{\frac{2}{3}}}\Big\}\times\\
& \Big\{1 - \frac{1}{E}(-c_1^2 + 12c_1c_2s^2 + 9c_2^2s^4 + 4sc_1^2)(b^2 - s^2) + \frac{2}{3E^4(c_1 + c_2s^2)}c_1s^3(144s^6c_1^3c_2\\
& - 79s^2c_1^4b^2 + 32s^3c_1^4b^2 + 72s^8c_1^2c_2^2 - 32b^4c_1^4s + 81b^4c_2^4s^8 + 16b^4s^2c_1^4 - 16b^2s^4c_1^4\\
& - 81b^2s^10c_2^4 + 72s^4c_1^4 + 7b^4c_1^4 + 162b^4c_1c_2^3s^6 + 72b^4c_1^3c_2s^3 + 72b^4c_2^2s^5c_1^2 - 72b^2s^7c_2^2c_1^2\\
& - 54s^4c_1^3b^2c_2 - 72s^6c_1^2b^2c_2^2 - 162s^8c_1c_2^3b^2 - 72s^5c_1^3c_2b^2 - 90b^4c_1^3c_2s^2)(5c_1^2 + 18c_1c_2s^2\\
& + 9c_2^2s^4 + 4sc_1^2)^2(b^2 - s^2)\Big\}
\end{aligned}$$

$$\begin{aligned}3p^2+6pq+(n+1)q^2=\frac{1}{E}(&12c_1^2(n+1)(4s^2c_1-b2c_1+3b2c_2s^2)^2+12c_1(4s^2c_1-b2c_1\\&+3b2c_2s^2)(27s^6c_2^2+27c_2^2s^5+18s^4c_1c_2+18c_1c_2s^3-17s^2c_1^2-9c_1^2s+8b2c_1^2)\\&+(27s^6c_2^2+27c_2^2s^5+18s^4c_1c_2+18c_1c_2s^3-17s^2c_1^2-9c_1^2s+8b2c_1^2)^2))\end{aligned}$$

We will prove that the 3-th root (α,β)-metric is Cartan bounded.

Proof of Theorem 1.1: First, suppose the dimension of the manifold is 2. local orthonormal coframe $\{\upsilon_1,\upsilon_2\}$ of Riemannian metric α. Therefore, α Can be written in the form of $\alpha^2=\upsilon_1^2+\upsilon_2^2$. We will do in the same way as in article [24]. If we adjust the coframe $\{\upsilon_1,\upsilon_2\}$ properly Thus $\beta=s\upsilon_1$.

Then $b_1=s$, $b_2=0$ where $\beta=\sum_{m=1}^{2}b_my^m$. Hence $\parallel\beta\parallel_\alpha:=\sqrt{a^{sm}b_sb_m}=r$.
For an arbitrary tangent vector $y=ve_1+ue_2\in TxM$, we can obtain that

$$\begin{aligned}&\alpha(x,y)=\sqrt{v^2+u^2},\qquad \beta=tv,\\&F(x,y)=\sqrt{v^2+u^2}(c_1\frac{tv}{\sqrt{v^2+u^2}}+c_2(\frac{tv}{\sqrt{v^2+u^2}})^3)^{\frac{1}{3}}.\end{aligned}$$

Assume that $y^\perp$ satisfies

$$\mathbf{g}_y(y,y^\perp)=0,\qquad \mathbf{g}_y(y^\perp,y^\perp)=F^2(x,y). \tag{3.1}$$

specifically Obviously $y^\perp$ is unique, because the metric is non-degenerate. The frame $\{y,y^\perp\}$ is called the Berwald frame.
Let $y=rcos\theta\mathbf{e}_1+rsin\theta\mathbf{e}_2$, i.e., $v=rcos\theta$ and $v=rsin\theta$. Putting the above expression into (3.1) and calculating with Maple (see Section A.1 in [24]) yields

$$y^\perp=(-\frac{\sqrt{2}rc_1\sin\theta\cos\theta}{\sqrt{c_1(4c_1\cos^2\theta+3t^2c_2\cos^2\theta-c_1)}},\frac{\sqrt{2}r(2c_1\cos^2\theta+3t^2c_2\cos^2\theta+c_1)}{2\sqrt{c_1(4c_1\cos^2\theta+3t^2c_2\cos^2\theta-c_1)}})$$

where

In [24],the norm of $\mathbf{C}$ is shown below

$$\parallel\mathbf{C}\parallel_x=\underset{y\in T_xM\setminus\{0\}}{sup}\eta(x,y),$$

where

$$\eta(x,y):=\frac{F(x,y)\mathbf{C}_y(y^\perp,y^\perp,y^\perp)}{|\mathbf{g}_y(y^\perp,y^\perp)|^{\frac{3}{2}}}.$$

By using Maple and Section A.2 in [24], we have

$$\eta(x,y)=\frac{\sqrt{2}}{2}|\frac{sin(\theta)(9c_2t^2\cos^2\theta+8c_1\cos^2\theta+c_1)rc_1}{(4c_1\cos^2\theta-c_1+3c_2t^2\cos^2\theta)\sqrt{c_1(4c_1\cos^2\theta-c_1+3c_2t^2\cos^2\theta)}}|$$

we define

$$f(t,z):=(4c_1z^2-c_1+3c_2t^2z^2)\sqrt{c_1(4c_1z^2-c_1+3c_2t^2z^2)}$$

$$g(t,z)=\frac{\sqrt{2}}{2f(t,z)}\sqrt{1-z^2}(9c_2t^2z^2+8c_1z^2+c_1)rc_1$$

where f and g are functions on $[0,1)\times[-1,1]$. Therefore

$$\parallel\mathbf{C}\parallel_x=\underset{0\leq\theta\leq2\pi}{max}|g(t,cos\theta)|.$$

We calculate

$$\lim_{t\longrightarrow 1^-} g(t,z)=\frac{\sqrt{2}}{2}\left|\frac{\sqrt{1-z^2}(9c_2z^2+8c_1z^2+c_1)rc_1}{(4c_1z^2-c_1+3c_2z^2)\sqrt{c_1(4c_1z^2+3c_2z^2-c_1)}}\right| \tag{3.2}$$

for $z=0$, we have

$$\lim_{t\longrightarrow 1^-} g(t,z)=\frac{\sqrt{2}}{2}\left|\frac{c_1^2}{-c_1\sqrt{-c_1^2}}\right| \tag{3.3}$$

where, $\sqrt{-c_1^2}$ undefined. $\lim_{t\longrightarrow 1^-} g(t,z)$ is not continuous for all $z\in[-1,1]$, so it is not bounded. This completes the proof. □

4. Conclusion

Among the class of m-th root Finsler metrics, 4-th root metrics have an important and special role in Finsler geometry. In this study, we focused on the class of 3-th root (α,β)-metrics and the bound Cartan torsion for a 3-th, 4-th and 5-th root (α,β)–metrics. We proved an important theorem about the necessary condition of unbounded Catan torsion.

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