THE SOLUTION OF THE TIME-FRACTIONAL DIFFUSION EQUATION BY THE VIETAFIBONACCI COLLOCATION AND RESIDUAL POWER SERIES METHODS

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Abstract.

In this paper, the numerical solution of the initial-value problem involving the time-fractional diffusion problem in the Caputo sense can be express as a series of the shifted Vieta-Fibonacci polynomials with unknown coefficients. Next, by making use of the collocation points and the relations between their coefficients via the boundary conditions, the recent problem is reduced to a system of fractional ordinary differential equations (SFODEs) with initial conditions. Then, utilizing the residual power series method (RPSM) on SFODEs, the analytic approximate solution can be achieved. To illustrate the simplicity and accuracy of the proposed method, some numerical examples are considered.

1. Introduction

The mathematical modelling of the wide range of problems in scientific and engineering fields appear as the time-fractional diffusion equations (TFDEs), see e.g. [1]-[4] and references therein. The TFDEs have been investigated in analytical and numerical frames by a number of authors, see e.g. [5]-[11].
In 2020, Bayrak et al. in [12] proposed a numerical solution for the TFDE using Chebyshev collocation method and the residual power series method (RPSM). In this paper, with the same idea, an efficient mathematical technique is successfully applied to obtain the analytical approximate solution of the TFDE,

$$D_t^\alpha u(x,t) = k(x) u_{xx}(x,t) + f(x,t), \quad 0 < \alpha \leq 1, \ 0 < x < l, \ 0 < t \leq T \tag{1.1}$$

subjected to the following initial-boundary conditions

$$u(x,0) = \phi(x), \quad 0 < x < l \tag{1.2}$$

$$u(0,t) = \mu_1(t) \ , \ u(l,t) = \mu_2(t) \quad 0 < t \leq T \tag{1.3}$$

where D_t^α denotes the Caputo fractional derivative of order $0 < \alpha \leq 1$. The diffusion coefficient $k(x)$, the source function $f(x,t)$ and also, $\phi(x)$, $\mu_1(t)$ and $\mu_2(t)$ are known functions.

2. Preliminaries

In this section, the Caputo fractional derivative (of order $\alpha \geq 0$) of $u(x,t)$ and also, the Vieta–Fibonacci polynomials will be reviewed.

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Definition 2.1. The α^{th} order Liouville-Caputo time-fractional derivative of $u(x,t)$ is defined as [13]

$$D_t^{\alpha}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-s)^{m-\alpha-1}\frac{\partial^m u(x,s)}{\partial s^m}ds, & m-1<\alpha<m,\\[2ex] \dfrac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\in\mathbb{N}.\end{cases}$$

Further, the Caputo fractional derivative of t^k $(k\geq 0)$ is given by

$$D_t^{\alpha}t^k=\begin{cases}\dfrac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}t^{k-\alpha}, & k\geq\lceil\alpha\rceil,\\[2ex] 0, & k<\lceil\alpha\rceil.\end{cases}$$

Definition 2.2. A power series of the form

$$\sum_{m=0}^{\infty} f_m(t-t_0)^{m\alpha}=f_0+f_1(t-t_0)^{\alpha}+f_2(t-t_0)^{2\alpha}+\cdots,\quad n-1<\alpha\leq n,\ t\geq t_0,$$

is called the fractional power series about $t=t_0$.

Theorem 2.3. *[13] Suppose that f has a fractional power series representation about $t=t_0$ of the form*

$$f(t)=\sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha},\quad n-1<\alpha\leq n,\ t_0\leq t<t_0+R.$$

If $D_t^{m\alpha}f(t)$, $(m=0,1,2,\ldots)$ are continuouse on (t_0,t_0+R), with R as the radius of convergence, then the coefficients c_m are given by formula

$$c_m=\frac{D_t^{m\alpha}f(t_0)}{\Gamma(m\alpha+1)},\quad m=0,1,2,\ldots,$$

where $D_t^{m\alpha}=\underbrace{D_t^{\alpha}D_t^{\alpha}\ldots D_t^{\alpha}}_{m-times}$.

Remark 2.4. Suppose that the solution of problem

$$D_t^{\alpha}u(x,t)+L[x]u(x,t)+N[x]u(x,t)=g(x,t),\quad 0<\alpha\leq 1,\quad 0\leq x\leq l,\ 0\leq t<T$$
$$u(x,0)=f_0(x),$$

with the linear and nonlinear operators $L[x],N[x]$,respectivley, as fracional power series expansion

$$u(x,t)=\sum_{n=0}^{\infty} f_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

Let

$$u_k(x,t)=\sum_{n=0}^{k} f_n(x)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\tag{2.1}$$

and

$$Res_k(x,t)=D_t^{\alpha}u_k(x,t)+L[x]u_k(x,t)+N[x]u_k(x,t)-g(x,t).$$

Then, the coffecients $f_n(x)$ in (2.1) can be determinid by the $RPSM$ as

$$D_t^{(n-1)\alpha}Res_n(x,t)|_{t=0}=0,\quad n=1,2,\ldots,k.\tag{2.2}$$

Definition 2.5. [14] The Vieta-Fibonacci polynomials of degree $n \in \mathbb{N}$ in x, $VF_n(x)$, are defined on $[-2, 2]$ as

$$VF_n(x) = \frac{\sin n\theta}{\sin\theta},$$

where $x = 2\cos\theta,\ \theta \in [0, \pi]$.

Remark 2.6. The analytical form of the $VF_n(x)$ can be given as follows:

$$VF_n(x) = \sum_{i=0}^{[\frac{n-1}{2}]} \frac{(-1)^i \Gamma(n-i)}{\Gamma(i+1)\Gamma(n-2i)} x^{n-2i-1}, \quad n \in \mathbb{N}.$$

Definition 2.7. The Shifted Vieta-Fibonacci polynomials of degree $n \in \mathbb{N}$ in x, $VF_n^*(x)$, are defined on $[0, l]$ as

$$VF_n^*(x) = VF_n(\frac{4}{l}x - 2).$$

Remark 2.8. The analytical form of the $VF_n^*(x)$ can be given as follows:

$$VF_n^*(x) = \sum_{i=0}^{n-1} \frac{(-1)^{n-i-1}(2)^{2i}\Gamma(n+i+1)}{\Gamma(n-i)\Gamma(2i+2)\, l^i} x^i, \quad n \in \mathbb{N}, \quad x \in (0, l).$$

3. The proposed method

The m^{th} degree approximation solution $u_m(x,t)$ of the problem (1.1)-(1.3) via $VF_{i+1}^*(x)$ $(i = 0, 1, \ldots, m)$ with the unkown coefficients c_{i+1}, can be constructed as follows:

$$u_m(x,t) = \sum_{i=0}^{m} c_{i+1}(t)\, VF_{i+1}^*(x) = C^T(t)\Phi(x), \tag{3.1}$$

where

$$C(t) = [c_1(t), c_2(t), \ldots, c_{m+1}(t)]^T\ ,\ \Phi(x) = [VF_1^*(x), VF_2^*(x), \ldots, VF_{m+1}^*(x)]^T$$

In order to acheive the solution $u_m(x,t)$ by Vieta-Fibonacci collocation method, the following steps are taken:

Step 1. Compute the collocation points $\{\mathbf{r}_j\}_{j=1}^{m}$ as the roots of $VF_{m+1}^*(x) = 0$.

Step 2. Construct a system of fractional ordinary differential equations (SFODEs) as follows:

$$\sum_{i=0}^{m} D_t^{\alpha} c_{i+1}(t) VF_{i+1}^*(r_j) = k(r_j) \sum_{i=0}^{m} c_{i+1}(t) \frac{d^2\, VF_{i+1}^*}{dx^2}(r_j) + f(r_j, t),\ j = 1, \ldots, m. \tag{3.2}$$

Step 3. Construct an algebraic system (AS) using initial and boundary conditions via the collocation points $\{\mathbf{r}_j\}_{j=1}^{m}$ as follows:

$$\begin{aligned} u_m(r_j, 0) &= \textstyle\sum_{i=0}^{m} c_{i+1}(0) VF_{i+1}^*(r_j) = \phi(r_j), \\ u_m(0,t) &= \textstyle\sum_{i=0}^{m} (-1)^i (i+1) c_{i+1}(t) = \mu_1(t), \\ u_m(l,t) &= \textstyle\sum_{i=0}^{m} (i+1) c_{i+1}(t) = \mu_2(t). \end{aligned} \tag{3.3}$$

Step 4. Utilizing the RPSM to solve the SFODEs (3.2) together with AS (3.3) in coefficients $c_{i+1}(t)$.

Step 5. Plugging $c_{i+1}(t),\ (i = 0, \ldots, m)$ into (3.1) to approximate the numerical solution $u_m(x,t)$.

4. Numerical experiments

In this section, by following the Steps 1-5, some of the time-fractional diffusion problems are numerically solved to check out the simplicity and effectiveness of the presented method.

Example 1. Consider the following problem

$$D_t^{\alpha}u(x,t)=\frac{1}{2}x^2u_{xx}(x,t),\quad 0<\alpha\leq 1,\ 0<x<1,\ 0<t<1 \tag{4.1}$$

$$u(x,0)=x^2, \tag{4.2}$$

$$u(0,t)=0\ ,\ u(1,t)=E_{\alpha}(t^{\alpha}) \tag{4.3}$$

where $E_{\alpha}(z)$ is the classical Mittag-Leffler function which is given by

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ Re(\alpha)>0$$

The exact solution of the problem (4.1) -(4.3) is given by $u(x,t)=x^2E_{\alpha}(t^{\alpha})$.

Case I. If $m=2$, then $r_1=\frac{1}{2}$. The SFODEs (3.2) and the AS (3.3) imply

$$D_t^{\alpha}c_1(t)-D_t^{\alpha}c_3(t)=4c_3(t), \tag{4.4}$$

$$c_1(0)-c_3(0)=\frac{1}{4}, \tag{4.5}$$

$$c_1(t)-2c_2(t)+3c_3(t)=0, \tag{4.6}$$

$$c_1(2)+2c_2(t)+3c_3(t)=E_{\alpha}(t^{\alpha}). \tag{4.7}$$

The Eqs. (4.6) and (4.7) imply

$$c_2(t)=\frac{1}{4}E_{\alpha}(t^{\alpha})\quad ,\quad c_3(t)=-\frac{1}{3}c_1(t)+\frac{1}{6}E_{\alpha}(t^{\alpha}) \tag{4.8}$$

Therefore, from the problem (4.4)-(4.5) with (4.8), the following initial value probelm can be obtained.

$$D_t^{\alpha}c_1(t)+c_1(t)-\frac{5}{8}E_{\alpha}(t^{\alpha})=0, \tag{4.9}$$

$$c_1(0)=\frac{5}{16}. \tag{4.10}$$

In this stage, for solving the problem (4.9) -(4.10) by the RPSM, letting

$$c_{10}(t)=f_0=\frac{5}{16}\quad ,\quad c_{11}(t)=f_0+f_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

imply

$$Res_1(t)=f_1+f_0+f_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}-\frac{5}{8}E_{\alpha}(t^{\alpha}),$$

and therefore, according to $Res_1(0)=0$, $f_1=\frac{5}{16}$.
Similarly, letting

$$c_{12}(t)=f_0+f_1\frac{t^{\alpha}}{\Gamma(\alpha+1)}+f_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

implies

$$Res_2(t) = f_1 + f_2\frac{t^\alpha}{\Gamma(\alpha+1)} + f_0 + f_1\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{5}{8}E_\alpha(t^\alpha).$$

Therefore, according to $D_t^\alpha Res_2(t)|_{t=0} = 0$, $f_2 = \frac{5}{16}$.
In a similar way, one obtains

$$f_3 = f_4 = \ldots = \frac{5}{16}.$$

So, from (2.1) we have $c_1(t) = \frac{5}{16}E_\alpha(t^\alpha)$, and from (4.8) it follows that

$$c_2(t) = \frac{1}{4}E_\alpha(t^\alpha) \quad , \quad c_3(t) = \frac{1}{16}E_\alpha(t^\alpha) \tag{4.11}$$

Finially, plugging (4.11) into (3.1) for $m = 2$ implies

$$u_2(x,t) = x^2 E_\alpha(t^\alpha),$$

which is the exact solution of the problem (4.1)-(4.3).

Example 2. Consider the following time-fractional diffusion equation ($0 < \alpha \leq 1$),

$$D_t^\alpha u(x,t) = -\frac{1}{2}x^2 u_{xx}(x,t) + (x^2 - x^3)t + (x^2 - 3x^3)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}, \quad 0 < x < 1,\ 0 < t < 1 \tag{4.12}$$

subjected to the following initial and boundary conditions

$$u(x,0) = 0, \tag{4.13}$$

$$u(0,t) = u(1,t) = 0. \tag{4.14}$$

The exact solution is given by $u(x,t) = x^2(1-x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$.
Case I. If $m = 2$, then $r_1 = \frac{1}{2}$ and so,

$$D_t^\alpha c_1(t) - D_t^\alpha c_3(t) = -4c_3(t) + \frac{1}{8}t - \frac{1}{8}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$c_1(0) - c_3(0) = 0,$$

$$c_1(t) - 2c_2(t) + 3c_3(t) = 0, \tag{4.15}$$

$$c_1(t) + 2c_2(t) + 3c_3(t) = 0. \tag{4.16}$$

From (4.15) and(4.16), it follows that

$$c_2(t) = 0 \quad , \quad c_3(t) = -\frac{1}{3}c_1(t)$$

Therefore,

$$D_t^\alpha c_1(t) - c_1(t) - \frac{3}{32}t + \frac{3}{32}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} = 0,$$

$$c_1(0) = 0. \tag{4.17}$$

or

$$D_t^{\alpha+1} c_1(t) - c_1'(t) - \frac{3}{32} + \frac{3}{32}\frac{t^\alpha}{\Gamma(\alpha+1)} = 0. \tag{4.18}$$

In this stage, for solving the problem (4.17) -(4.18) by the RPSM, letting

$$c_{10}(t) = f_0 = 0 \quad , \quad c_{11}(t) = f_0 + f_1 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

imply

$$Res_1(t) = f_1 - f_1 t^{\alpha} - \frac{3}{32} + \frac{3}{32}\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

Therefore, according to $Res_1(0) = 0$, we have $f_1 = \frac{3}{32}$.
Similarly, letting

$$c_{12}(t) = f_1 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + f_2 \frac{t^{2(\alpha+1)}}{\Gamma(2\alpha+3)},$$

implies

$$Res_2(t) = f_1 + f_2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - f_1 \frac{t^{\alpha}}{\Gamma(\alpha+1)} - f_2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{3}{32} + \frac{3}{32}\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

According to $D_t^{\alpha+1} Res_2(t)|_{t=0} = 0$, one obtains $f_2 = 0$.
In a similar way, from (4.15) and(4.16), it follows that $f_3 = f_4 = \ldots = 0$ and therefore,

$$c_1(t) = \frac{3}{32}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \quad , \quad c_2(t) = 0 \quad , \quad c_3(t) = -\frac{1}{32}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

Thus,

$$u_2(x,t) = \frac{1}{2}x(1-x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{4.19}$$

Case II. If $m = 3$, then $r_1 = \frac{1}{2}$, $r_2 = \frac{3}{4}$ and the following SFODEs can be obtained.

$$D_t^{\alpha} c_0 1(t) - D_t^{\alpha} c_3(t) + 8 D_t^{\alpha} c_4(t) = -4c_3(t) + \frac{1}{8}t - \frac{1}{8}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$D_t^{\alpha} c_1(t) + D_t^{\alpha} c_2(t) + 7 D_t^{\alpha} c_4(t) = -9c_3(t) + 27c_4(t) + \frac{9}{64}t - \frac{45}{64}\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

$$c_1(0) - c_3(0) + 8c_4(0) = 0,$$

$$c_1(0) + c_2(0) + 7c_4(0) = 0,$$

$$c_1(t) - 2c_2(t) + 3c_3(t) - 4c_4(t) = 0, \tag{4.20}$$

$$c_1(t) + 2c_2(t) + 3c_3(t) + 4c_4(t) = 0. \tag{4.21}$$

From (4.20)-(4.21) it follows that

$$c_3(t) = -\frac{1}{3}c_1(t) \quad , \quad c_4(t) = -\frac{1}{2}c_2(t)$$

and therefore,

$$D_t^{\alpha} c_1(t) - 0.03125 c_1(t) - 0.09375 t + 0.09375 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} = 0,$$

$$D_t^{\alpha} c_1(t) + 1.5 D_t^{\alpha} c_2(t) - 0.09375 c_1(t) - 13.5 c_2(t) - 0.140625 t + 0.703125 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} = 0,$$

$$c_1(0) = 0 \quad , \quad c_2(0) = 0 \tag{4.22}$$

or

$$D_t^{\alpha+1}c_1(t) - 0.03125c_1'(t) - 0.09375 + 0.09375\frac{t^\alpha}{\Gamma(\alpha+1)} = 0, \tag{4.23}$$

$$D_t^{\alpha+1}c_1(t) + 1.5D_t^{\alpha+1}c_2(t) - 0.09375c_1'(t) - 13.5c_2'(t) - 0.140625 + 0.703125\frac{t^\alpha}{\Gamma(\alpha+1)} = 0. \tag{4.24}$$

In this stage, for solving the problem (4.23) -(4.24) with the initial conditions (4.22) by the RPSM, letting

$$c_{10}(t) = f_0 = 0 \quad , \quad c_{11} = f_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$
$$c_{20}(t) = g_0 = 0 \quad , \quad c_{21} = g_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

imply

$$Res_1(t) = \begin{cases} f_1 - 0.03125f_1\frac{t^\alpha}{\Gamma(\alpha+1)} - 0.09375 + 0.09375\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ \\ f_1 + 1.5g_1 - 0.09375f_1\frac{t^\alpha}{\Gamma(\alpha+1)} - 13.5g_1\frac{t^\alpha}{\Gamma(\alpha+1)} - 0.140625 + 0.703125\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{cases}$$

Applying the condition $Res_1(0) = 0$, it follows that $f_1 = 0.09375$ and $g_1 = 0.03125$.
Similarly, letting

$$c_{12}(t) = f_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + f_2\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)},$$
$$c_{22}(t) = g_1\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + g_2\frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)},$$

imply

$$Res_2(t) = \begin{cases} f_1 + f_2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - 0.03125\{f_1\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\} - 0.09375 + 0.09375\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ \\ f_1 + f_2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 1.5\{g_1 + g_2\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\} - 0.09375\{f_1\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\} \\ \qquad -13.5\{g_1\frac{t^\alpha}{\Gamma(\alpha+1)} + g_2\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 0.140625 + 0.703125\frac{t^\alpha}{\Gamma(\alpha+1)}\}. \end{cases}$$

Therefore,

$$D_t^{\alpha+1}Res_2(t) = \begin{cases} f_2 - 0.03125f_2\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ \\ f_2 + 1.5g_2 - 0.09375f_2\frac{t^\alpha}{\Gamma(\alpha+1)} - 13.5g_2\frac{t^\alpha}{\Gamma(\alpha+1)}. \end{cases}$$

Letting $D_t^{\alpha+1}Res_2(0) = 0$ implies $f_2 = g_2 = 0$.
Similarly, $f_3 = f_4 = \ldots = 0$ and $g_3 = g_4 = \ldots = 0$, and it follows that

$$c_1(t) = 0.09375\,\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \quad , \quad c_2(t) = 0.03125\,\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$
$$c_3(t) = -0.03125\,\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \quad , \quad c_4(t) = -0.015625\,\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

and then, the numerical solution of problem (4.12)-(4.14) can be obtained as follows

$$u_3(x,t) = x^2(1-x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)},$$

which is the exact solution.

5. Conclusions

In this paper, by following [12], the numerical solution of initial-boundary value problem involving the time-fractional diffusion equation has been achieved by the Vieta–Fibonacci collocation method and the residual power series method (RPSM). Simplicity and efficiency of the presented method have been considered in some of illustrative examples.

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