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SOME EXISTENCE AND UNIQUENESS RESULTS OF THE MATHEMATICAL MODELING FOR A PARTICULAR TYPE OF INFLUENZA VIRUS BY THE FRACTAL-FRACTIONAL DERIVATIVE

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Abstract.

This paper proposes a novel model for the dynamics of susceptible, exposed, infectious, and recovered individuals infected with the AH1N1/09 influenza strain. The model leverages fractal-fractional operators with power-law kernels to capture the inherent memory effects and complex transmission patterns associated with the virus. We establish existence criteria using two approaches to analyze the qualitative behavior of the model's solutions. Firstly, we demonstrate the existence of solutions through the concept of α-ψ contractions and α-admissible mappings. Secondly, we employ the Leray-Schauder theorem to provide an alternative existence condition. Finally, the uniqueness of the solution is investigated by exploiting the Lipschitz property.

1. Introduction

In recent years, novel influenza strains like SARS-CoV-2 have emerged, along with its associated variants (e.g., Delta, Omicron). Mathematical modeling provides a valuable tool for understanding the dynamics of such epidemics [1, 2].

Fractional calculus has gained traction in epidemic modeling due to its ability to capture inherent memory effects not fully represented by integer-order models. Fractional derivatives, such as the Caputo-Fabrizio and Atangana-Baleanu derivatives [3–6], have been successfully employed for this purpose.

Atangana (2017) introduced the concept of fractal-fractional operators, which bridge the gap between fractional and fractal calculus [8]. These operators combine power-law, exponential, and generalized Mittag-Leffler kernels with fractal derivatives, offering a more comprehensive framework for capturing complex real-world phenomena. The two key components of these operators are the fractional order and the fractal dimension.

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This paper leverages the advantages of fractal-fractional operators to formulate a novel mathematical model for the dynamics of the AH1N1/09 influenza virus. We believe this approach can provide a more accurate and insightful representation of the virus's transmission patterns.

2. Mathematical Background

This section establishes the mathematical framework for analyzing the SEIR model of the AH1N1/09 influenza virus using fractal-fractional operators. We introduce relevant concepts from fixed-point theory and fractal-fractional calculus.

2.1. **Increasing Operators.** Let Ψ denote a collection of increasing functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following properties:
* $\sum_{\jmath=1}^{\infty} \psi^{\jmath}(\mathrm{t}) < \infty$ for all $\mathrm{t} > 0$.
* $\psi(\mathrm{t}) < \mathrm{t}$ for all $\mathrm{t} > 0$.

2.2. **α-ψ-Contractions and α-Admissible Mappings.** Consider a normed space X and a function $\alpha : \mathrm{X}^2 \to \mathbb{R}^+ \cup \{0\}$. We define two properties for mappings $\mathbf{F} : \mathrm{X} \to \mathrm{X}$:
* α-ψ-contraction: A mapping $\mathbf{F}$ is an α-ψ-contraction if the following holds for all $x_1, x_2 \in \mathrm{X}$:

$$\alpha(x_1, x_2)\mathrm{d}(\mathbf{F}x_1, \mathbf{F}x_2) \leq \psi(d(x_1, x_2))$$

* α-admissible: A mapping $\mathbf{F}$ is α-admissible if the following implication holds:

$$\alpha(x_1, x_2) \geq 1 \Rightarrow \alpha(\mathbf{F}x_1, \mathbf{F}x_2) \geq 1$$

These concepts play a crucial role in establishing the existence of solutions for the fractal-fractional model.

2.3. **Fractal-Fractional Derivatives.** We introduce the concept of the fractal-fractional derivative, which combines elements of both fractal and fractional calculus.

Definition 2.1. Let $\mathrm{F} : (a,b) \to [0,\infty)$ be a continuous function that is fractal differentiable of order ν. The fractal-fractional derivative of F in the Riemann-Liouville sense with a power-law type kernel of order ω is defined as:

$$^{\mathrm{FFP}}\mathrm{D}_{a,\mathrm{t}}^{\omega,\nu}\mathrm{F}(\mathrm{t}) = \frac{1}{\Gamma(n-\omega)} \frac{\mathrm{d}}{\mathrm{dt}^{\nu}} \int_a^{\mathrm{t}} (\mathrm{t}-\mathrm{W})^{n-\omega-1}\mathrm{F}(\mathrm{W})\,\mathrm{dW}, \tag{2.1}$$

where $\dfrac{\mathrm{dF(W)}}{\mathrm{dW}^{\nu}} = \lim_{\mathrm{t}\to\mathrm{W}} \dfrac{\mathrm{F(t)} - \mathrm{F(W)}}{\mathrm{t}^{\nu} - \mathrm{W}^{\nu}}$ denotes the fractal derivative and $n-1 < \omega, \nu \leq n \in \mathbb{N}$.

As noted, when $\nu = 1$, the fractal-fractional derivative reduces to the standard ω^{th}-Riemann-Liouville derivative.

2.4. **Fractal-Fractional Integrals.** (Previously fixed missing closing curly brace)
We also define the concept of a fractal-fractional integral.

Definition 2.2. A continuous map F defined on (a,b) is said to be fractal-fractional integrable of fractional and fractal orders ω and ν respectively, via the power law type kernel, if the integral

$$^{\mathrm{FFP}}\mathfrak{I}_{a,\mathrm{t}}^{\omega,\nu}\mathrm{F}(\mathrm{t}) = \frac{\nu}{\Gamma(\omega)} \int_a^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathrm{F}(\mathrm{W})\,\mathrm{dW}, \tag{2.2}$$

exists, where $\nu, \omega > 0$.

2.5. **Generalized Caputo-Type Derivatives.** Here, we introduce the concept of the generalized Caputo-type derivative, another approach within the framework of fractal-fractional operators.

Definition 2.3. [10] The generalized ω^{th}-Liouville-Caputo-type (GLC-type) derivative ${}^{\mathbf{GLC}}\mathrm{D}^{\omega,\rho}_{a+}$ is presented as

$$({}^{\mathbf{GLC}}\mathrm{D}^{\omega,\rho}_{a+}\mathrm{F})(\mathrm{t}) = \frac{\rho^{\omega-n+1}}{\Gamma(n-\omega)}\int_a^{\mathrm{t}} \mathrm{W}^{\rho-1}(\mathrm{t}^\rho - \mathrm{W}^\rho)^{n-\omega-1}\Big(\mathrm{W}^{1-\rho}\frac{\mathrm{d}}{\mathrm{dW}}\Big)^n \mathrm{F}(\mathrm{W})\,\mathrm{dW},\ \mathrm{t} > a, \tag{2.3}$$

with $\rho > 0$ and $n-1 < \omega \leq n$.

We will utilize both power-law kernels and generalized Caputo-type derivatives within the $\mathcal{SEIR}$ model analysis to gain a comprehensive understanding of the virus dynamics.

3. Description of the AH1N1/09 model

One of the proposed models for virus AH1N1 is the $\mathcal{SEIR}$ model, which was designed by Gilberto Gonzalez Para and his *et al.* [7] in the framework of the ordinary derivative, and provided valuable results in the study of epidemic diseases. We divide the total population, denoted by the symbol $\mathcal{N}(\mathrm{t})$ in the present model, into four subclasses $\mathrm{H}(\mathrm{t})$, $\mathrm{I}(\mathrm{t})$, $\mathrm{L}(\mathrm{t})$ and $\mathrm{S}(\mathrm{t})$ at the time $\mathrm{t} \in \mathbb{J} := [0, T], (T > 0)$ which are the susceptible, exposed, infectious and recovered people. Hence, we formulate the mentioned $\mathcal{SEIR}$ model by

$$\begin{cases} \dfrac{\mathrm{dH(t)}}{\mathrm{dt}} = p - q\mathrm{H(t)L(t)} - r\mathrm{H(t)}, \\ \dfrac{\mathrm{dI(t)}}{\mathrm{dt}} = q\mathrm{H(t)L(t)} - (r+s)\mathrm{I(t)}, \\ \dfrac{\mathrm{dL(t)}}{\mathrm{dt}} = s\mathrm{I(t)} - (r+b)\mathrm{L(t)}, \\ \dfrac{\mathrm{dS(t)}}{\mathrm{dt}} = b\mathrm{L(t)} - r\mathrm{S(t)}, \end{cases} \tag{3.1}$$

where p and r are the birth and death rates of people, the rate q stands for the amount of transmission of infection from L to H, and the rate s shows the amount of transmission of people from I to L. Also, the recovery rate is illustrated by the parameter b at the time $\mathrm{t} \in \mathbb{J} := [0, T], (T > 0)$. The relevant initial conditions for above model are

$$\mathrm{H}(0) = \mathrm{H}_0,\ \mathrm{I}(0) = \mathrm{I}_0,\ \mathrm{L}(0) = \mathrm{L}_0,\ \mathrm{S}(0) = \mathrm{S}_0,$$

where $\mathrm{H}_0, \mathrm{I}_0, \mathrm{L}_0 > 0$, and $\mathrm{S}_0 \geq 0$.

Motivated by the above standard model, we here consider the fractal-fractional model of the AH1N1/09 virus in the following structure:

$$\begin{cases} {}^{\mathrm{FFP}}\mathrm{D}^{\omega,\nu}_{0,\mathrm{t}}\mathrm{H(t)} = p - q\mathrm{H(t)L(t)} - r\mathrm{H(t)}, \\ {}^{\mathrm{FFP}}\mathrm{D}^{\omega,\nu}_{0,\mathrm{t}}\mathrm{I(t)} = q\mathrm{H(t)L(t)} - (r+s)\mathrm{I(t)}, \\ {}^{\mathrm{FFP}}\mathrm{D}^{\omega,\nu}_{0,\mathrm{t}}\mathrm{L(t)} = s\mathrm{I(t)} - (r+b)\mathrm{L(t)}, \\ {}^{\mathrm{FFP}}\mathrm{D}^{\omega,\nu}_{0,\mathrm{t}}\mathrm{S(t)} = b\mathrm{L(t)} - r\mathrm{S(t)}, \end{cases} \tag{3.2}$$

with the initial values $\mathrm{H}_0, \mathrm{I}_0, \mathrm{L}_0, \mathrm{S}_0$ for the above four state functions, where ${}^{\mathrm{FFP}}\mathrm{D}^{\omega,\nu}_{0,\mathrm{t}}$ is the fractal-fractional derivative with the fractional order $\omega \in (0, 1]$ and the fractal order $\nu \in (0, 1]$ via the power law type kernel. We impose some important assumptions on our model. All parameters are nonnegative and the state functions give $\mathcal{N}(\mathrm{t}) = \mathrm{H}(\mathrm{t}) + \mathrm{I}(\mathrm{t}) + \mathrm{L}(\mathrm{t}) + \mathrm{S}(\mathrm{t})$ at the time $\mathrm{t} \in \mathbb{J} := [0, T], (T > 0)$.

4. Existence of solutions

Before analyzing and investigating any given biological mathematical model, it is common to control whether in fact the mentioned dynamical system exists or not. The answer of such a question is guaranteed with the help of fixed point theory. For this, we aim to utilize the same theory for the suggested fractal-fractional system (3.2) being part of the present study. For our study, we regard the Banach space $\mathrm{X} = \mathbb{Y}^4$ with $\mathbb{Y} = C(\mathbb{J}, \mathbb{R})$ and

$$\|\mathbb{W}\|_{\mathrm{X}} = \|\big(\mathrm{H}, \mathrm{I}, \mathrm{L}, \mathrm{S}\big)\|_{\mathrm{X}} = \max\big\{|\mathrm{H}(\mathrm{t})| + |\mathrm{I}(\mathrm{t})| + |\mathrm{L}(\mathrm{t})| + |\mathrm{S}(\mathrm{t})| : \ \mathrm{t} \in \mathbb{J}\big\}.$$

Next, the R.H.S. of the fractal-fractional model of the AH1N1/09 virus (3.2) can be rewritten as:

$$\begin{cases} \mathbb{G}_1\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) = p - q\mathrm{H}(\mathrm{t})\mathrm{L}(\mathrm{t}) - r\mathrm{H}(\mathrm{t}), \\ \mathbb{G}_2\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) = q\mathrm{H}(\mathrm{t})\mathrm{L}(\mathrm{t}) - (r+s)\mathrm{I}(\mathrm{t}), \\ \mathbb{G}_3\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) = s\mathrm{I}(\mathrm{t}) - (r+b)\mathrm{L}(\mathrm{t}), \\ \mathbb{G}_4\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) = b\mathrm{L}(\mathrm{t}) - r\mathrm{S}(\mathrm{t}). \end{cases} \tag{4.1}$$

Because of the differentiability of integral, we can reformulate the suggested fractal-fractional model of the AH1N1/09 virus (3.2) as:

$$\begin{cases} {}^{\mathbf{RL}}\mathrm{D}^{\omega}_{0,\mathrm{t}}\mathrm{H}(\mathrm{t}) = \nu\mathrm{t}^{\nu-1}\mathbb{G}_1\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big), \\ {}^{\mathbf{RL}}\mathrm{D}^{\omega}_{0,\mathrm{t}}\mathrm{I}(\mathrm{t}) = \nu\mathrm{t}^{\nu-1}\mathbb{G}_2\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big), \\ {}^{\mathbf{RL}}\mathrm{D}^{\omega}_{0,\mathrm{t}}\mathrm{L}(\mathrm{t}) = \nu\mathrm{t}^{\nu-1}\mathbb{G}_3\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big), \\ {}^{\mathbf{RL}}\mathrm{D}^{\omega}_{0,\mathrm{t}}\mathrm{S}(\mathrm{t}) = \nu\mathrm{t}^{\nu-1}\mathbb{G}_4\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big). \end{cases} \tag{4.2}$$

By the system (4.2) ($\forall\, \mathrm{t} \in \mathbb{J}$), the developed system can be represented in the compact model of the following IVP

$$\begin{cases} {}^{\mathbf{RL}}\mathrm{D}^{\omega}_{0,\mathrm{t}}\mathbb{W}(\mathrm{t}) = \nu\mathrm{t}^{\nu-1}\mathbb{G}\big(\mathrm{t}, \mathbb{W}(\mathrm{t})\big), \ \omega, \nu \in (0, 1], \\ \mathbb{W}(0) = \mathbb{W}_0, \end{cases} \tag{4.3}$$

where

$$\mathbb{W}(\mathrm{t}) = \big(\mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big)^T, \ \mathbb{W}_0 = \big(\mathrm{H}_0, \mathrm{I}_0, \mathrm{L}_0, \mathrm{S}_0\big)^T, \tag{4.4}$$

and

$$\mathbb{G}\big(\mathrm{t}, \mathbb{W}(\mathrm{t})\big) = \begin{cases} \mathbb{G}_1\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big), \\ \mathbb{G}_2\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big), \\ \mathbb{G}_3\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big), \\ \mathbb{G}_4\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big). \end{cases} \tag{4.5}$$

Further, we operate on both sides of the compact form (4.3) by the fractal-fractional integral which is given by (2.2), and we find that

$$\mathbb{W}(\mathrm{t}) = \mathbb{W}(0) + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathbb{G}\big(\mathrm{W}, \mathbb{W}(\mathrm{W})\big)\,d\mathrm{W}. \tag{4.6}$$

In other words, the extended form of the above fractal-fractional integral equation is represented as the following system

$$\begin{cases} \mathrm{H}(\mathrm{t}) = \mathrm{H}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathbb{G}_1\big(\mathrm{W}, \mathrm{H}(\mathrm{W}), \mathrm{I}(\mathrm{W}), \mathrm{L}(\mathrm{W}), \mathrm{S}(\mathrm{W})\big)\,d\mathrm{W}, \\ \mathrm{I}(\mathrm{t}) = \mathrm{I}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathbb{G}_2\big(\mathrm{W}, \mathrm{H}(\mathrm{W}), \mathrm{I}(\mathrm{W}), \mathrm{L}(\mathrm{W}), \mathrm{S}(\mathrm{W})\big)\,d\mathrm{W}, \\ \mathrm{L}(\mathrm{t}) = \mathrm{L}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathbb{G}_3\big(\mathrm{W}, \mathrm{H}(\mathrm{W}), \mathrm{I}(\mathrm{W}), \mathrm{L}(\mathrm{W}), \mathrm{S}(\mathrm{W})\big)\,d\mathrm{W}, \\ \mathrm{S}(\mathrm{t}) = \mathrm{S}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathbb{G}_4\big(\mathrm{W}, \mathrm{H}(\mathrm{W}), \mathrm{I}(\mathrm{W}), \mathrm{L}(\mathrm{W}), \mathrm{S}(\mathrm{W})\big)\,d\mathrm{W}. \end{cases} \tag{4.7}$$

In this place, we transform the initial fractal-fractional problem (3.2) into the fixed point problem. Define $G : \mathrm{X} \to \mathrm{X}$ by

$$G(\mathbb{W}(\mathrm{t})) = \mathbb{W}(0) + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1}\mathbb{G}\big(\mathrm{W}, \mathbb{W}(\mathrm{W})\big)\,d\mathrm{W}. \tag{4.8}$$

In the sequel, we recall the required fixed point theorem in connection with our aim for proving the existence results.

Theorem 4.1. *[9] Let (X, d) be a complete metric space, $\alpha : \mathrm{X} \times \mathrm{X} \to \mathbb{R}$, $\psi \in \Psi$ and $\mathbb{G} : \mathrm{X} \to \mathrm{X}$ be an α-ψ-contractive map. Assume that*

(1) *$\mathbb{G}$ is α-admissible on X;*
(2) *for some $x_0 \in \mathrm{X}$, $\alpha(x_0, \mathbb{G}x_0) \geq 1$;*
(3) *for any sequence $\{x_n\}$ in X with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$, $\forall\, n \geq 1$, we have $\alpha(x_n, x) \geq 1$, $\forall\, n \geq 1$.*

Then $\exists x^* \in \mathrm{X}$ s.t. $\mathbb{G}x^* = x^*$.

In the first place, the first existence criterion is proved here under some special operators.

Theorem 4.2. *Let $\exists\, \xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\exists\, \mathbb{G} \in C(\mathbb{J} \times \mathrm{X}, \mathrm{X})$ and $\exists\, \psi \in \Psi$ such that:*

($\mathfrak{P}_1$): $\forall\, \mathbb{W}_1, \mathbb{W}_2 \in \mathrm{X}$ *and* $\mathrm{t} \in \mathbb{J}$,

$$|\mathbb{G}(\mathrm{t}, \mathbb{W}_1(\mathrm{t})) - \mathbb{G}(\mathrm{t}, \mathbb{W}_2(\mathrm{t}))| \leq \tilde{\ell}\, \psi(|\mathbb{W}_1(\mathrm{t}) - \mathbb{W}_2(\mathrm{t}|),$$

with $\xi\big(\mathbb{W}_1(\mathrm{t}), \mathbb{W}_2(\mathrm{t})\big) \geq 0$, *where* $\tilde{\ell} = \dfrac{\Gamma(\nu+\omega)}{\nu T^{\nu+\omega-1}\Gamma(\nu)}$;

($\mathfrak{P}_2$): $\exists\, \mathbb{W}_0 \in \mathrm{X}$ *so that* $\forall\, \mathrm{t} \in \mathbb{J}$,

$$\xi\big(\mathbb{W}_0(\mathrm{t}), G(\mathbb{W}_0(\mathrm{t}))\big) \geq 0,$$

and also $\xi\big(\mathbb{W}_1(\mathrm{t}), \mathbb{W}_2(\mathrm{t})\big) \geq 0$ *gives*

$$\xi\big(G(\mathbb{W}_1(\mathrm{t})), G(\mathbb{W}_2(\mathrm{t}))\big) \geq 0,$$

$\forall\, \mathbb{W}_1, \mathbb{W}_2 \in \mathrm{X}$ *and* $\mathrm{t} \in \mathbb{J}$;

($\mathfrak{P}_3$): $\forall \{\mathbb{W}_n\}_{n\geq 1} \subseteq \mathrm{X}$ *with* $\mathbb{W}_n \to \mathbb{W}$ *and*

$$\xi\big(\mathbb{W}_n(\mathrm{t}), \mathbb{W}_{n+1}(\mathrm{t})\big) \geq 0, \ (\forall\, n \in \mathbb{N}, \forall\, \mathrm{t} \in \mathbb{J}),$$

we get

$$\xi\big(\mathbb{W}_n(\mathrm{t}), \mathbb{W}(\mathrm{t})\big) \geq 0.$$

Then, it is found a solution for fractal-fractional problem (4.3)*, and so it is found a solution to the given fractal-fractional epidemic model of AH1N1/09 virus* (3.2)*.*

Proof. To begin the proof, we select $\mathbb{W}_1$ and $\mathbb{W}_2$ as two elements of X arbitrarily with the following property

$$\xi\big(\mathbb{W}_1(\mathrm{t}), \mathbb{W}_2(\mathrm{t})\big) \geq 0,$$

for each $\mathrm{t} \in \mathbb{J}$. In this case, by definition of the Beta function, we may write

$$\begin{aligned}
\big|G(\mathbb{W}_1(\mathrm{t})) - G(\mathbb{W}_2(\mathrm{t}))\big| &\leq \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1} \big|\mathbb{G}\big(\mathrm{W}, \mathbb{W}_1(\mathrm{W})\big) - \mathbb{G}\big(\mathrm{W}, \mathbb{W}_2(\mathrm{W})\big)\big|\, \mathrm{dW} \\
&\leq \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t}-\mathrm{W})^{\omega-1} \tilde{\ell}\psi\big(|\mathbb{W}_1(\mathrm{W}) - \mathbb{W}_2(\mathrm{W}|\big)\, \mathrm{dW} \\
&\leq \frac{\nu\tilde{\ell}T^{\nu+\omega-1}\mathbb{B}(\nu,\omega)}{\Gamma(\omega)} \psi\big(\|\mathbb{W}_1 - \mathbb{W}_2\|_{\mathrm{X}}\big) \\
&= \frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)} \tilde{\ell}\psi\big(\|\mathbb{W}_1 - \mathbb{W}_2\|_{\mathrm{X}}\big).
\end{aligned}$$

Consequently, we have

$$\|G(\mathbb{W}_1) - G(\mathbb{W}_2)\|_{\mathrm{X}} \leq \frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)} \tilde{\ell}\psi\big(\|\mathbb{W}_1 - \mathbb{W}_2\|_{\mathrm{X}}\big) = \psi\big(\|\mathbb{W}_1 - \mathbb{W}_2\|_{\mathrm{X}}\big).$$

We make the nonnegative map $\alpha : \mathrm{X} \times \mathrm{X} \to [0, \infty)$ which takes the form

$$\alpha(\mathbb{W}_1, \mathbb{W}_2) = \begin{cases} 1 \text{ if } \xi\big(\mathbb{W}_1(\mathrm{t}), \mathbb{W}_2(\mathrm{t})\big) \geq 0, \\ 0 \text{ otherwise}, \end{cases}$$

for every $\mathbb{W}_1, \mathbb{W}_2 \in \mathrm{X}$. Then, for any $\mathbb{W}_1, \mathbb{W}_2 \in \mathrm{X}$, it is obtained

$$\alpha(\mathbb{W}_1, \mathbb{W}_2)\mathrm{d}(G(\mathbb{W}_1), G(\mathbb{W}_2)) \leq \psi(\mathrm{d}(\mathbb{W}_1, \mathbb{W}_2)).$$

Thus, G is α-ψ-contraction. To see that G is α-admissible, let $\mathbb{W}_1, \mathbb{W}_2 \in \mathrm{X}$ with $\alpha(\mathbb{W}_1, \mathbb{W}_2) \geq 1$. By definition of α, it gives

$$\xi\big(\mathbb{W}_1(\mathrm{t}), \mathbb{W}_2(\mathrm{t})\big) \geq 0.$$

By ($\mathfrak{P}_2$), $\xi\big(G(\mathbb{W}_1(\mathrm{t})), G(\mathbb{W}_2(\mathrm{t}))\big) \geq 0$ is satisfied. Once again by applying definition of α, we get $\alpha(G(\mathbb{W}_1), G(\mathbb{W}_2)) \geq 1$, which confirms that G is α-admissible.

As the condition ($\mathfrak{P}_2$) ensures the existence of $\mathbb{W}_0 \in \mathrm{X}$, thus for every $\mathrm{t} \in \mathbb{J}$,

$$\xi\big(\mathbb{W}_0(\mathrm{t}), G(\mathbb{W}_0(\mathrm{t}))\big) \geq 0,$$

which is clearly deduced that $\alpha(\mathbb{W}_0, G(\mathbb{W}_0)) \geq 1$. These results confirms that the conditions (1) and (2) of Theorem 4.1 are fulfilled.

In the sequel of the proof, let $\{\mathbb{W}_n\}_{n\geq 1} \subseteq \mathrm{X}$ be a sequence with $\mathbb{W}_n \to \mathbb{W}$ and $\alpha(\mathbb{W}_n, \mathbb{W}_{n+1}) \geq 1$ for each n. Definition of the non-negative map α gives

$$\xi\big(\mathbb{W}_n(\mathrm{t}), \mathbb{W}_{n+1}(\mathrm{t})\big) \geq 0.$$

Hence $(\mathfrak{P}_3)$ implies that

$$\xi\big(\mathbb{W}_n(\mathrm{t}), \mathbb{W}(\mathrm{t})\big) \geq 0.$$

This means that $\alpha(\mathbb{W}_n, \mathbb{W}) \geq 1$, $\forall\, n$, and shows that the condition (3) of Theorem 4.1 is fulfilled. Lastly, the conclusion of Theorem 4.1 ensures the existence of some fixed point for G named as $\mathbb{W}^* \in \mathrm{X}$. This means that $\mathbb{W}^* = \big(\mathrm{H}^*, \mathrm{I}^*, \mathrm{L}^*, \mathrm{S}^*\big)^T$ is a solution to the fractal-fractional model of AH1N1/09 virus (3.2) and the argument is finally ended. □

5. Uniqueness result

To prove the uniqueness of solution of the given fractal-fractional model of AH1N1/09 virus (3.2), we use the Lipschitz property of kernel functions $\mathbb{G}_i$, $(i = 1, \dots, 4)$ given by (4.1).

Lemma 5.1. *Let the functions* $\mathrm{H}, \mathrm{I}, \mathrm{L}, \mathrm{S}, \mathrm{H}^*, \mathrm{I}^*, \mathrm{L}^*, \mathrm{S}^* \in \mathbb{Y} := C(\mathbb{J}, \mathbb{R})$. *Let*

(H1) $\|\mathrm{H}\| \leq \lambda_1$, $\|\mathrm{I}\| \leq \lambda_2$, $\|\mathrm{L}\| \leq \lambda_3$, $\|\mathrm{S}\| \leq \lambda_4$ *for some constants* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$.

Then the kernels $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_3, \mathbb{G}_4$ *introduced by* (4.1) *are satisfied the Lipschitz property w.r.t. the corresponding components if* $w_1, w_2, w_3, w_4 < 1$, *where*

$$w_1 = q\lambda_3 + r, \ \ w_2 = r + s,$$
$$w_3 = r + b, \ \ w_4 = r. \tag{5.1}$$

Proof. We begin with the kernel function $\mathbb{G}_1$. For each $\mathrm{H}, \mathrm{H}^* \in \mathbb{Y} := C(\mathbb{J}, \mathbb{R})$, we have

$$\begin{aligned}
&\|\mathbb{G}_1\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) - \mathbb{G}_1\big(\mathrm{t}, \mathrm{H}^*(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big)\| \\
&\qquad = \Big\|\big(p - q\mathrm{H}(\mathrm{t})\mathrm{L}(\mathrm{t}) - r\mathrm{H}(\mathrm{t})\big) - \big(p - q\mathrm{H}^*(\mathrm{t})\mathrm{L}(\mathrm{t}) - r\mathrm{H}^*(\mathrm{t})\big)\Big\| \\
&\qquad \leq \big[q\|\mathrm{L}(\mathrm{t})\| + r\big]\|\mathrm{H}(\mathrm{t}) - \mathrm{H}^*(\mathrm{t})\| \\
&\qquad \leq \big[q\lambda_3 + r\big]\|\mathrm{H}(\mathrm{t}) - \mathrm{H}^*(\mathrm{t})\| \\
&\qquad = w_1\|\mathrm{H}(\mathrm{t}) - \mathrm{H}^*(\mathrm{t})\|.
\end{aligned}$$

This shows that $\mathbb{G}_1$ is Lipschitz w.r.t. H with constant $w_1 < 1$. Regarding the kernel function $\mathbb{G}_2$, for each $\mathrm{I}, \mathrm{I}^* \in \mathbb{Y} := C(\mathbb{J}, \mathbb{R})$, we have

$$\begin{aligned}
&\|\mathbb{G}_1\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) - \mathbb{G}_1\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}^*(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big)\| \\
&\qquad = \Big\|\big(q\mathrm{H}(\mathrm{t})\mathrm{L}(\mathrm{t}) - (r+s)\mathrm{I}(\mathrm{t})\big) - \big(q\mathrm{H}(\mathrm{t})\mathrm{L}(\mathrm{t}) - (r+s)\mathrm{I}^*(\mathrm{t})\big)\Big\| \\
&\qquad \leq \big[r + s\big]\|\mathrm{I}(\mathrm{t}) - \mathrm{I}^*(\mathrm{t})\| \\
&\qquad = w_2\|\mathrm{I}(\mathrm{t}) - \mathrm{I}^*(\mathrm{t})\|.
\end{aligned}$$

This shows that $\mathbb{G}_2$ is Lipschitz w.r.t. I with constant $w_2 < 1$. Now, for each $\mathrm{L}, \mathrm{L}^* \in \mathbb{Y} := C(\mathbb{J}, \mathbb{R})$, we have

$$\begin{aligned}
&\|\mathbb{G}_3\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) - \mathbb{G}_3\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}^*(\mathrm{t}), \mathrm{S}(\mathrm{t})\big)\| \\
&\qquad = \Big\|\big(s\mathrm{I}(\mathrm{t}) - (r+b)\mathrm{L}(\mathrm{t})\big) - \big(s\mathrm{I}(\mathrm{t}) - (r+b)\mathrm{L}^*(\mathrm{t})\big)\Big\| \\
&\qquad \leq \big[r + b\big]\|\mathrm{L}(\mathrm{t}) - \mathrm{L}^*(\mathrm{t})\| \\
&\qquad = w_3\|\mathrm{L}(\mathrm{t}) - \mathrm{L}^*(\mathrm{t})\|.
\end{aligned}$$

Accordingly, this shows that $\mathbb{G}_3$ is Lipschitz w.r.t. L with constant $w_3 < 1$. Now, for each $\mathrm{S}, \mathrm{S}^* \in \mathbb{Y} := C(\mathbb{J}, \mathbb{R})$, we have

$$\begin{aligned}
&\|\mathbb{G}_4\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}(\mathrm{t})\big) - \mathbb{G}_4\big(\mathrm{t}, \mathrm{H}(\mathrm{t}), \mathrm{I}(\mathrm{t}), \mathrm{L}(\mathrm{t}), \mathrm{S}^*(\mathrm{t})\big)\| \\
&\qquad = \big\|\big(b\mathrm{L}(\mathrm{t}) - r\mathrm{S}(\mathrm{t})\big) - \big(b\mathrm{L}(\mathrm{t}) - r\mathrm{S}^*(\mathrm{t})\big)\big\| \\
&\qquad \le \big[r\big]\|\mathrm{S}(\mathrm{t}) - \mathrm{S}^*(\mathrm{t})\| \\
&\qquad = w_4\|\mathrm{S}(\mathrm{t}) - \mathrm{S}^*(\mathrm{t})\|.
\end{aligned}$$

This shows that $\mathbb{G}_4$ is Lipschitz w.r.t. S with constant $w_4 < 1$. Above results confirm that four kernel functions $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_3, \mathbb{G}_4$ are Lipschitzian w.r.t. the corresponding component with constants w_1, w_2, w_3, w_4, respectively. ☐

Based on the obtained results in Lemma 5.1, we investigate the uniqueness criterion for solution to the supposed fractal-fractional system (3.2).

Theorem 5.2. *Let (H1) holds. Then the given fractal-fractional model of AH1N1/09 virus* (3.2) *has a unique solution if*

$$\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)} w_i < 1, \;\; i \in \{1, \ldots, 4\}. \tag{5.2}$$

Proof. We assume that the conclusion of theorem is not valid. In other words, another solution exists for the given fractal-fractional model of AH1N1/09 virus (3.2). Assume that $\big(\mathrm{H}^*(\mathrm{t}), \mathrm{I}^*(\mathrm{t}), \mathrm{L}^*(\mathrm{t}), \mathrm{S}^*(\mathrm{t})\big)$ is another solution with initial conditions $\big(\mathrm{H}_0, \mathrm{I}_0, \mathrm{L}_0, \mathrm{S}_0\big)$ provided that by (4.7), we have

$$\mathrm{H}^*(\mathrm{t}) = \mathrm{H}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t} - \mathrm{W})^{\omega-1} \mathbb{G}_1\big(\mathrm{W}, \mathrm{H}^*(\mathrm{W}), \mathrm{I}^*(\mathrm{W}), \mathrm{L}^*(\mathrm{W}), \mathrm{S}^*(\mathrm{W})\big)\, d\mathrm{W},$$

$$\mathrm{I}^*(\mathrm{t}) = \mathrm{I}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t} - \mathrm{W})^{\omega-1} \mathbb{G}_2\big(\mathrm{W}, \mathrm{H}^*(\mathrm{W}), \mathrm{I}^*(\mathrm{W}), \mathrm{L}^*(\mathrm{W}), \mathrm{S}^*(\mathrm{W})\big)\, d\mathrm{W},$$

$$\mathrm{L}^*(\mathrm{t}) = \mathrm{L}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t} - \mathrm{W})^{\omega-1} \mathbb{G}_3\big(\mathrm{W}, \mathrm{H}^*(\mathrm{W}), \mathrm{I}^*(\mathrm{W}), \mathrm{L}^*(\mathrm{W}), \mathrm{S}^*(\mathrm{W})\big)\, d\mathrm{W},$$

and

$$\mathrm{S}^*(\mathrm{t}) = \mathrm{S}_0 + \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t} - \mathrm{W})^{\omega-1} \mathbb{G}_4\big(\mathrm{W}, \mathrm{H}^*(\mathrm{W}), \mathrm{I}^*(\mathrm{W}), \mathrm{L}^*(\mathrm{W}), \mathrm{S}^*(\mathrm{W})\big)\, d\mathrm{W}.$$

Now, we can estimate

$$\begin{aligned}
|\mathrm{H}(\mathrm{t}) - \mathrm{H}^*(\mathrm{t})| &\le \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t} - \mathrm{W})^{\omega-1} \\
&\quad \times \big|\mathbb{G}_1\big(\mathrm{W}, \mathrm{H}(\mathrm{W}), \mathrm{I}(\mathrm{W}), \mathrm{L}(\mathrm{W}), \mathrm{S}(\mathrm{W})\big) - \mathbb{G}_1\big(\mathrm{W}, \mathrm{H}^*(\mathrm{W}), \mathrm{I}^*(\mathrm{W}), \mathrm{L}^*(\mathrm{W}), \mathrm{S}^*(\mathrm{W})\big)\big|\, d\mathrm{W} \\
&\le \frac{\nu}{\Gamma(\omega)} \int_0^{\mathrm{t}} \mathrm{W}^{\nu-1}(\mathrm{t} - \mathrm{W})^{\omega-1} w_1\|\mathrm{H} - \mathrm{H}^*\|\, d\mathrm{W} \\
&\le \frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)} w_1\|\mathrm{H} - \mathrm{H}^*\|,
\end{aligned}$$

and so

$$\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_1\right]\|\mathrm{H}-\mathrm{H}^*\|\leq 0.$$

The latter inequality is true if $\|\mathrm{H}-\mathrm{H}^*\|=0$, and accordingly $\mathrm{H}=\mathrm{H}^*$. Similarly, from

$$\|\mathrm{I}-\mathrm{I}^*\|\leq\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_2\right]\|\mathrm{I}-\mathrm{I}^*\|,$$

we get

$$\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_2\right]\|\mathrm{I}-\mathrm{I}^*\|\leq 0.$$

This implies that $\|\mathrm{I}-\mathrm{I}^*\|=0$ and so $\mathrm{I}=\mathrm{I}^*$. Also,

$$\|\mathrm{L}-\mathrm{L}^*\|\leq\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_3\right]\|\mathrm{L}-\mathrm{L}^*\|.$$

This gives

$$\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_3\right]\|\mathrm{L}-\mathrm{L}^*\|\leq 0.$$

Hence $\mathrm{L}=\mathrm{L}^*$. Finally, from

$$\|\mathrm{S}-\mathrm{S}^*\|\leq\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_4\right]\|\mathrm{S}-\mathrm{S}^*\|,$$

we get

$$\left[1-\frac{\nu T^{\nu+\omega-1}\Gamma(\nu)}{\Gamma(\nu+\omega)}w_4\right]\|\mathrm{S}-\mathrm{S}^*\|\leq 0.$$

So $\mathrm{S}=\mathrm{S}^*$. Consequently, we get

$$\big(\mathrm{H(t)},\mathrm{I(t)},\mathrm{L(t)},\mathrm{S(t)}\big)=\big(\mathrm{H}^*(\mathrm{t}),\mathrm{I}^*(\mathrm{t}),\mathrm{L}^*(\mathrm{t}),\mathrm{S}^*(\mathrm{t})\big).$$

This shows that the fractal-fractional model of AH1N1/09 virus (3.2) has a unique solution and this ends our proof. □

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