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INVESTIGATING THE LIE SYMMETRY METHOD FOR GENERALIZED MODIFIED HEAT EQUATION

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Abstract.

Employing Lie group analysis, we aim to systematically explore the inherent symmetries of the generalized modified heat equation. This analysis provides valuable insights into the invariance properties, unveiling hidden structures that can contribute to a deeper understanding of the equation. Through the scrutiny of Lie symmetries, our objective is to facilitate the identification of exact solutions for the generalized modified heat equation. This contributes to the development of analytical methods and enhances our ability to comprehend the physical implications of the solutions.

1. Introduction

Lie theory was pioneered by mathematician Sophus Lie. Caprasse and Élie Cartan have evolved into a powerful mathematical framework for understanding the symmetries inherent in differential equations. The Lie group and Lie algebra concepts provide a profound basis for analyzing and solving differential equations by revealing the transformations that leave equations invariant. The contributions of Olver and Bluman have significantly enriched the field of Lie theory, particularly in its applications to differential equations [1, 2]. Ian Olver, in his influential work "Applications of Lie Groups to Differential Equations," has provided a comprehensive overview of the theory's applications in various branches of mathematics and physics. Olver's insights have been instrumental in connecting the abstract concepts of Lie groups to practical problem-solving techniques for differential equations. Additionally, George W. Bluman, in collaboration with Shulim Kalitzin and Allen F. Cheviakov, has made substantial contributions to the development of the Lie symmetry method. The book "Symmetry and Integration Methods for Differential Equations," coauthored by Bluman and Cheviakov, serves as a valuable resource for researchers and practitioners seeking a deeper understanding of Lie symmetries and their applications. Bluman's work has played a pivotal role in advancing the application of Lie symmetries to solve differential equations in various scientific disciplines. Lie group method are widely employed for solving large classes of differential equations and they stand out as one of the most powerful tools in this domain. For instance, a multitude of diverse problems have been successfully addressed using these techniques, as evidenced by the references [3–11]. Partial differential equations (PDEs) play a pivotal role in modeling a wide range of physical phenomena, offering insights into the behavior of complex systems. Recently, researchers have proposed and used various effective methods to solve equations within the realm of nonlinear dynamics. Among these methods are the G'/G expansion method [12–14]. , the extended hyperbolic-function method [15–17], and the Kudryashov method [18–20]. These techniques have proven to be instrumental in providing analytical solutions for a wide range of equations. Mohanty et al. [21] employed the G'/G-expansion method to construct solitary waves of the

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mKdV equation. Hosseini et al. [22]found dark solitons of a nonlinear Schrödinger equation with parabolic law using the Kudryashov method. In [23] Asjad at al. utilizing the extended hyperbolic-function method to obtain optical solitons of Biswas-Arshed equation. One of the most important and widely used partial differential equations is heat equation. The heat equation is a fundamental partial differential equation that describes the distribution of heat (or temperature) over time in a given region. The heat equation and generalized it are extensively used in various fields of science and engineering to model heat transfer phenomena [24–27].
In this study, we focus on generalized modified heat equation that exhibits intriguing nonlinear characteristics, aiming to explore its symmetry properties through the lens of Lie group analysis.
The generalized modified heat equation under consideration is given by:

$$u_t + (u+\alpha)u_{xx} + \beta u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t)$. while the terms involving u_t, u_x, u_{xx}, and u_{xxx} signify temporal and spatial derivatives. The coefficients α and β introduce additional complexity to the equation. The overarching objective of this research is to employ Lie group analysis, a powerful mathematical tool, to investigate the symmetry transformations inherent in the given PDE. Symmetry analysis provides valuable insights into the invariance properties of a differential equation under certain transformations, revealing hidden structures and facilitating the identification of exact solutions. Through a systematic examination of the Lie symmetries, we aim to gain a deeper understanding of the underlying dynamics governed by this nonlinear PDE. This investigation may open avenues for the development of analytical methods and shed light on the physical significance of the solutions.

2. Lie symmetry analysis of governing equation

In the current section, we deal with Eq. (1.1) and its Lie symmetry analysis. To this end, we construct Lie group transformations with $\mathcal{E}$ as a group parameter as follows

$$\begin{aligned}
&x^* \longmapsto x + \mathcal{E}\varsigma^x(x,t,u) + \mathcal{O}\left(\mathcal{E}^2\right),\\
&t^* \longmapsto t + \mathcal{E}\varsigma^t(x,t,u) + \mathcal{O}\left(\mathcal{E}^2\right),\\
&u^* \longmapsto u + \mathcal{E}\varsigma^u(x,t,u) + \mathcal{O}\left(\mathcal{E}^2\right),
\end{aligned} \tag{2.1}$$

where ς^x, ς^t, and ς^u are the infinitesimal generators. The corresponding Lie algebra is given by

$$X = \varsigma^x(x,t,u)\frac{\partial}{\partial x} + \varsigma^t(x,t,u)\frac{\partial}{\partial t} + \varsigma^u(x,t,u)\frac{\partial}{\partial u}.$$

For Eq. (1.1), such a vector field creates symmetry if and only if

$$\mathcal{P}r^{(3)}X(\mathcal{F})|_{\mathcal{F}=0} = 0, \tag{2.2}$$

where $\mathcal{P}r^{(3)}(X)$is the third prolongation of $X$for Eq. (1.1). Owing to Eq. (2.2), the following invariance condition can be established

$$\mathcal{P}r^{(3)}X\left(u_t + (u+\alpha)u_{xx} + \beta u_{xxx}\right)|_{(1)} = 0.$$

Using the extended operator of $\mathcal{P}r^{(3)}(X)$ we have

$$\mathcal{P}r^{(3)}X(\mathcal{F}) = u_{xx}\varsigma^u + \varsigma^u_t + (u+\alpha)\varsigma^u_{xx} + \beta\varsigma^u_{xxx},$$

and invariant condition is derived as

$$u_{xx}\varsigma^u + \varsigma^u_t + (u+\alpha)\varsigma^u_{xx} + \beta\varsigma^u_{xxx} = 0,$$

where ς^u, ς^u_t, ς^u_{xx}, and ς^u_{xxx} are the coefficients of $\mathcal{P}r^{(3)}(X)$. By equating the coefficients of u_x, u_t, u_{xx}, u_x^2, uu_{xx}, $\cdots$ to zero, the following system is acquired

$$\begin{aligned}
&\varsigma^x_u = \varsigma^t_u = \varsigma^t_x = 0,\\
&\varsigma^u_{uu} = 0, 3\varsigma^x_x - \varsigma^t_t = 0,\\
&(u+\alpha)\varsigma^u_{xx} + \beta\varsigma^u_{xxx} + \varsigma^u_t = 0,\\
&3\beta\varsigma^u_{xxu} - (u+\alpha)\,\varsigma^x_{xx} + 2\,(u+\alpha)\,\varsigma^u_{xu} - \beta\varsigma^x_{xxx} - \varsigma^x_t = 0,\\
&3\beta\varsigma^u_{xu} + \varsigma^u - 2\,(u+\alpha)\,\varsigma^x_x + (u+\alpha)\,\varsigma^t_t - 3\beta\varsigma^x_{xx} = 0.
\end{aligned}$$

Due to the above system, vector fields corresponding to Eq. (1.1) are constructed as

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, \qquad X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - (u+\alpha)\,\frac{\partial}{\partial u}.$$

Among these vector fields becomes

Table.1: commutator table for the vector fields of Eq. (1.1)

$[X_i,\ X_j]$	X_1	X_2	X_3
X_1	0	0	$3X_1$
X_2	0	0	X_2
X_3	$-3X_1$	$-X_2$	0

In Lie symmetry method, a commutator table is a systematic way to determine the commutation relations between the vector fields associated with the symmetries of a differential equation. This table helps in understanding how the symmetries interact with each other when applied sequentially to a given equation. To construct a commutator table, one typically considers pairs of vector fields associated with the symmetries of the equation. The commutator of two vector fields, denoted as $[X_i, X_j]$, is calculated by applying one vector field after the other and subtracting the result from applying the vector fields in the opposite order. Mathematically, this is expressed as

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

When dealing with any subalgebra of these symmetries, one can achieve reduction through characteristic equations

$$\frac{dx}{\varsigma^x} = \frac{dt}{\varsigma^t} = \frac{du}{\varsigma^u},$$

and we write the solutions of Eq. (1.1) according to the above vector fields as follows:

Case (I): Solutions of Eq. (1.1) via the use of vector field X_1

For the vector field $X_1 = \frac{\partial}{\partial x}$, we have following characteristic equations

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0},$$

and the invariant solution for this case is $u\,(x,\ t) = h\,(x)$. So, the simple ordinary differential equation is derived as

$$(h\,(x) + \alpha)\,h^{''}\,(x) + \beta h'''(x) = 0.$$

Thus, the exact solution of this equation is $h\,(x) = ax + b$, where a and b are constants. Therefore, the exact solution of Eq. (1.1) is $u\,(x,t) = ax + b$.

Case (II): Solutions of Eq. (1.1) via the use of vector field X_2

For the vector field $X_2 = \frac{\partial}{\partial t}$, characteristic equations as follows

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0},$$

so, the invariant solution is $u(x,t) = h(t)$. The reduced equation owing to such an invariant solution is

$$h'(t) = 0.$$

Thus, $h(t) = c$ and the trivial exact solution of Eq. (1.1) is $u(x,t) = c$, where c is arbitrary constant.

Case (III): Solutions of Eq. (1.1) via the use of vector field X_3

For the vector field $X_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - (u+\alpha)\frac{\partial}{\partial u}$, the characteristic equations

$$\frac{dx}{x} = \frac{dt}{3t} = \frac{du}{-(u+\alpha)},$$

and the invariant solution is $u(x,t) = -\alpha + \frac{1}{x}h(\xi)$, where $\xi = \frac{t}{x^3}$. The reduced equation is

$$\begin{aligned} &-27\beta h''(\xi)\xi^3 + 9\xi^2 h(\xi)h''(\xi) - 135\beta h''(\xi)\xi^2 + 18\xi h(\xi)h'(\xi) - 114\beta\xi h'(\xi) + 2h^2(\xi) \\ &-6\beta h(\xi) + h'(\xi) = 0. \end{aligned}$$

The exact solution related to this case is

$$u(x,t) = -\alpha + 3\beta c_1 \tanh\left(2\beta c_1^3 t + c_1 x + c_2\right).$$

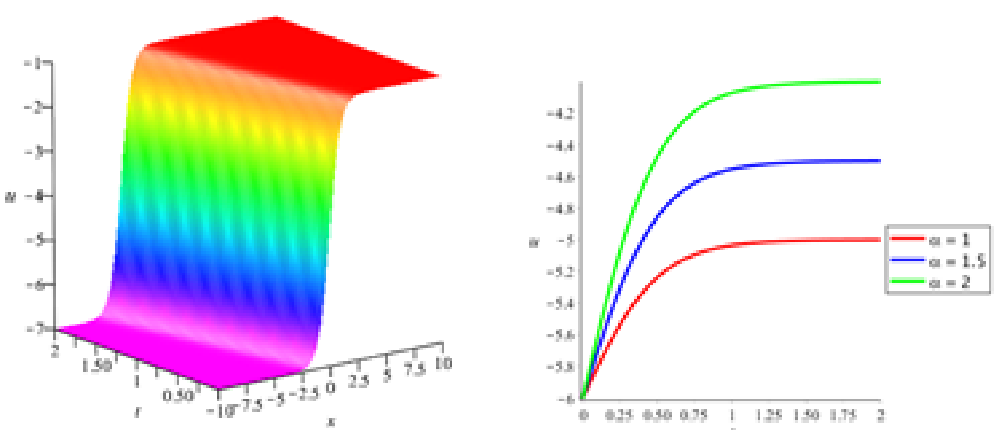

Figure 1: Plots related to solutions of Eq. (1.1) derived by classical vector field X_3 (a) 3D-plot for $\alpha = 2, \ \beta = c_1 = 1$ and $c_2 = 0$, (b) 2D-plot for $\beta = c_1 = 1$, $x = c_2 = 0$,and $\alpha = 1, \alpha = 1.5, \alpha = 2$.

3. Conclusion

In summary, using Lie symmetries and associated Lie point generators is a powerful approach for studying partial differential equations. The systematic application of Lie group methods reveals hidden symmetries in PDEs, leading to the derivation of Lie point generators and determining equations. Solving these determining equations determines symmetry group transformations and invariant solutions. This method elegantly transitions from PDEs to coupled ordinary differential equations through Lie symmetry analysis. The invariant solutions from these ODEs serve as foundational components for constructing solutions to the original PDEs. This approach simplifies the analysis of complex PDEs and provides a systematic method for transforming them into ODEs, often more amenable to analytical or numerical solutions. The interplay between Lie symmetries Lie point generators, and invariant solutions enhances our understanding of the underlying structures of differential equations, providing deeper insights into the behavior of systems described by these equations.

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