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ON A CLASS OF WEAKLY LANDSBERG 4-TH ROOT FINSLER METRICS

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Abstract.

The theory of m-th root Finsler metrics is used in Ecology, Biology, Seismic ray theory, Gravitation, etc. It is introduced as a direct generalization of Riemannian metric, in other words, the second root metric is a Riemannian metric. On the other hand, the Riemannian curvature actually reveals the local geometric properties of a Riemann–Finsler metric. In this work, we investigate the class of 4-th root (α, β)-metrics. We prove that every weakly Landsberg 4-th root (α, β)-metrics has vanishing S-curvature. By employing it, we show that a 4-th root (α, β)-metric is a weakly Landsberg metric if and only if it is a Berwald metric.

1. Introduction

Consider a Finsler metric $F = F(x, y)$ on a manifold M of dimension n. Let $G^s = G^s(x, y)$ denote the spray coefficients of F in a local coordinate system.The Landsbergc curvature $\mathbf{L} = L_{stu}(x, y)dx^s \otimes dx^t \otimes dx^u$ is a horizontal on $TM/0$, defined by

$$L_{stu} := -\frac{1}{2} F F_{y^k} [G^k]_{y^s y^t y^u}.$$

Finsler metrics F are called Landsberg metrics if $L_{stu} = 0$ .The mean Landsbergc curvature $\mathbf{J} = J_i dx^i$, defined by

$$J_k := g^{st} L_{stk}$$

If $\mathbf{J} = 0$, Finsler metrics F are weakly Landsberg metrics. obviously, in dimension 2, any weakly Landsberg metric must be a Landsberg metric [1]. In [2], Tayebi and Izadian show that every weakly Landsberg fourth root (α, β)-metric on an $n \geq 3$-dimensional manifold M is a Berwald metric (assuming that $s_{ij} = 0$). Here, without considering $s_{ij} = 0$, we show every weakly Landsberg 4-th root (α, β)-metric on a manifold of dimension n is Berwald metric(see [3–7]). First, we calculate the S-curvature for the 4-th root (α, β)-meter and use it to show the following theorem

Theorem 1.1. *Consider $F = \sqrt[4]{c_1\alpha^4 + c_2\alpha^2\beta^2 + c_3\beta^4}$ be a fourth root (α, β)-metric on a Finsler manifold of dimension n. Then F is Weakly Landsberg metric if and only if $r_{ij} = 0$, $s_{ij} = 0$ (that is,β is parallel with respect to α).*

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2. Preliminaries

For a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}, \quad y \in T_x M. \tag{2.1}$$

$\mathbf{G}$ is called the associated spray to (M, F) [8].

The function $F = \alpha\phi(s)$ is a Finsler metric ($\alpha = \sqrt{a_{ts} y^t y^s}$ and $\beta = b_m y^m$ where $||\beta_x||_\alpha < b_0$) if and only if ϕ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition:

$$\phi''(s)(b^2 - s^2) - s\phi'(s) + \phi(s) > 0, \qquad |s| \le b < b_0. \tag{2.2}$$

Obviously, $\phi = \phi(s)$ must satisfy

$$\phi(s) - \phi'(s)s > 0, \qquad |s| < b_0.$$

The expressed metric $F = \alpha\phi(s)$ on a manifold M is called an (α, β)-metric. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$r^i{}_j := a^{is} r_{sj}, \quad s^i{}_j := a^{is} s_{sj}, \quad q_{ij} := r_{is} s^s{}_j, \quad t_{ij} := t_{ik} s^k{}_j,$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}, \quad q_j := b^i q_{ij}, \quad t_j := b^i t_{ij},$$

where ”|” denotes the covariant derivative with respect to the Levi-Civita connection of α and $b^i := a^{ij} b_j$, a^{ij} is the inverse of a_{ij}. We define $r_{i0} = r_{ij} y^j$ and $r_{00} = r_{ij} y^i y^j$, etc [9]. For a function $\phi = \phi(s)$ satisfying (2.2). Put

$$\begin{aligned}
Q :=& \frac{\phi'}{\phi - s\phi'},\\
\Delta :=& 1 + sQ + (b^2 - s^2)Q',\\
\Psi :=& \frac{\phi''}{2[(B - s^2)\phi'' + (\phi - s\phi')]},\\
\Theta :=& \frac{\phi'\phi - (\phi'\phi' + \phi''\phi)s}{2\phi((\phi - s\phi') + (B - s^2)\phi''},
\end{aligned}$$

where $B := ||\beta||_\alpha^2$. Let $G^t = G^t(x, y)$ and $G^t_\alpha = G^t_\alpha(x, y)$ denote the coefficients of F and α, respectively, in the same coordinate system. We have

$$G^t = G^t_\alpha + \alpha Q s^t{}_0 + (r_{00} - 2Q\alpha s_0)(\alpha^{-1}\Theta y^t + \Psi b^t). \tag{2.3}$$

where

$$P := \Big[-2Q\alpha s_0 + r_{00} \Big] \Theta \alpha^{-1}, \quad Q^t := \Psi \Big[r_{00} - 2\alpha Q s_0 \Big] b^t + \alpha Q s^t{}_0.$$

Obviously, if $r_{ij} = 0$ and $s_{ij} = 0$ (β is parallel with respect to α), then $P = 0$ and $Q^i = 0$.]n other words, F is a Berwald metric. Let

$$\Phi := (sQ' - Q)\{n\Delta + Qs + 1\} - Q''(sQ + 1)(B - s^2).$$

We can obtain a formula for the mean Cartan torsion of (α,β)- metrics as

$$I_j=-\frac{(\phi-s\phi')\Phi}{2\Delta\phi\alpha^2}(\alpha b_j-sy_j). \tag{2.4}$$

Thus $\mathbf{I}=0$ if and only if $\Phi=0$.

3. Proof of Theorem 1.1

In [10], Abazari-Khoshdani characterized weakly Berwald fourth root metrics in the form of $F^4=c_1\alpha^4+c_2\alpha^2\beta^2+c_3\beta^4$ (c_1, c_2 and c_3 are constants).

Assume F be an (α,β)-metric on an n-dimensional manifold M. Then the S-curvature of F is given by

$$\mathbf{S}=\Big[2\Psi-\frac{f'(b)}{bf(b)}\Big](s_0+r_0)-\frac{\Phi}{2\Delta^2\alpha}(r_{00}-2Q\alpha s_0), \tag{3.1}$$

where

$$f(b):=\frac{\int_0^\pi(b\cos t)T\ \sin^{n-2}t\ dt}{\int_0^\pi\sin^{n-2}tdt},\qquad T(s):=(\phi-\phi's)^{n-2}\phi\big[\phi''(b^2-s^2)(\phi-\phi's)\big].$$

Here, we calculate the S-curvature of 5-th root (α,β)-metric and obtain the following.

Lemma 3.1. *The S-curvature of 4-th root (α,β)-metric is given by*

$$\begin{aligned}
\mathbf{S}\ =\ &\frac{1}{AB}\Big\{c_2c_1+c_2^2s^2+7c_2c_3s^4+6c_3s^2c_1+6c_3^2s^6-\frac{f'(b)}{bf(b)}\Big\}\big(s_0+r_0\big)\\
&-\frac{1}{2\alpha A^2B^3}\Big\{s(16s^5nb^2c_2^2c_1^2c_3+12s^4nc_1^2c_2c_3-8s^6nc_1c_2^2c_3-40s^8nc_1c_2c_3^2+4s^3nb^2c_2^3c_1^2\\
&-8s^5nb^2c_2^4c_1+56s^9nb^2c_2^4c_3+300s^{11}nb^2c_2^3c_3^2+768s^{13}nb^2c_2^2c_3^3+936s^{15}nb^2c_2c_3^4\\
&+16s^5nb^2c_3^2c_1^3-48s^9nb^2c_3^3c_1^2-16s^7nc_2^2c_1^2c_3+80s^9nc_2^3c_1c_3+272s^{11}nc_2^2c_1c_3^2\\
&-8s^5nc_2c_1^3c_3+8s^9nc_2c_1^2c_3^2+360s^{13}nc_2c_1c_3^3-144s^{13}nb^2c_3^4c_1-38b^2c_1^2c_3c_2s^2\\
&-44b^2c_1c_3c_2^2s^4-140b^2c_1c_2s^6c_3^2+44s^4c_3c_1^2c_2+40s^6c_2^2c_1c_3+120s^8c_3^2c_2c_1+4s^6nc_1^2c_3^2\\
&-36s^{10}nc_1c_3^3-20s^8nc_2^3c_3-66s^{10}nc_2^2c_3^2-84s^{12}nc_2c_3^3+4s^7nb^2c_2^5+432s^{17}nb^2c_3^5\\
&-4s^5nc_2^3c_1^2+8s^7nc_2^4c_1-56s^{11}nc_2^4c_3-300s^{13}nc_2^3c_3^2-768s^{15}nc_2^2c_3^3-936s^{17}nc_2c_3^4\\
&-16s^7nc_3^2c_1^3+48s^{11}nc_3^3c_1^2+144s^{15}nc_3^4c_1+2s^2nc_1^2c_2^2+4s^2nc_1^3c_3-90b^2c_1c_3^3s^8-4b^2c_2^3s^2\\
&-78b^2c_1^2c_3^2s^4-54b^2c_2s^{10}c_3^3-10b^2c_2^3s^6c_3-45b^2c_2^2s^8c_3^2+8s^2c_3c_1^3+4s^2c_2^2c_1^2+80s^6c_3^2c_1^2\\
&+72s^{10}c_3^3c_1+12s^{12}c_3^3c_2+12s^{10}c_3^2c_2^2-2s^6nc_2^4-36s^{14}nc_3^4-4s^9nc_2^5-432s^{19}nc_3^5-6b^2c_3\\
&-3b^2c_1^2c_2^2-b^2c_2^4s^4-18b^2c_3^4s^{12}+4s^4c_1c_2^3-80s^7nb^2c_2^3c_1c_3-272s^9nb^2c_2^2c_1c_3^2\\
&+8s^3nb^2c_2c_1^3c_3-8s^7nb^2c_2c_1^2c_3^2-360s^{11}nb^2c_2c_1c_3^3)\Big\}\Big(Br_{00}+4s(c_2+2c_3s^2)s_0\Big),
\end{aligned}$$

where

$$\begin{aligned}
A:&=-c_1-c_2s^2-c_3s^4-2sb^2c_2c_1+2b^2c_2^2s^3+10b^2c_2c_3s^5-4b^2c_3s^3c_1+12b^2c_3^2s^7+2c_2s^3c_1\\
&\quad-2c_2^2s^5-10c_2s^7c_3+4c_3s^5c_1-12c_3^2s^9,\\
B:&=-c_1+c_2s^2+3c_3s^4,\\
T:&=(c_1+c_2s^2+c_3s^4)^2(c_1+c_2s^2+c_3s^4)^{n-2}.
\end{aligned}$$

Now, we study weakly Landsberg 4-th root (α,β)-metrics and show the following.

Theorem 3.2. *Every weakly Landsberg 4-th root (α, β)-metric has vanishing S-curvature.*

Proof. For an (α, β)-metric $F = \alpha\phi(s)$, the mean Landsberg curvature is given by

$$J_t = -\frac{1}{2\Delta\alpha^4}\Bigg[\frac{2\alpha^2}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](r_0+s_0)h_t + \frac{\alpha}{b^2-s^2}(\Psi_1+s\frac{\Phi}{\Delta})(r_{00}-2\alpha Q s_0)h_t + \alpha\Big[-\alpha Q' s_0 h_t + \alpha Q(\alpha^2 s_t - y_t s_0) + \alpha^2\Delta s_{i0} + \alpha^2(r_{i0}-2\alpha Q s_t) - (r_{00}-2\alpha Q s_0)y_t\Big]\frac{\Phi}{\Delta}\Bigg], \tag{3.2}$$

where

$$\Psi_1 := \sqrt{b^2-s^2}\Big[\frac{\sqrt{b^2-s^2}\Phi}{\Delta^{\frac{3}{2}}}\Big]'\Delta^{\frac{1}{2}}, \quad h_t := b_t - \alpha^{-1} s y_t.$$

contracting (3.2) with b^t and simplifying it, we have $\mathbf{J} = b^t J_t = 0$ [1]. It is equal to following

$$d_6\alpha^6 + d_5\alpha^5 + d_4\alpha^4 + d_2\alpha^2 + d_0 = 0, \tag{3.3}$$

where

$$\begin{aligned}
d_0 :&= -216\beta^6 r_{00} c_3^3,\\
d_2 :&= 1728\beta^3 c_3^3 b^2 s_0 - 648\beta^2 c_3^3 b^4 r_{00} - 3264\beta^3 c_3^2 s_0 c_2 + 576\beta^2 c_3^2 r_{00} c_1 - 3006\beta^2 c_3 r_{00} c_2^2\\
&\quad - 48 c_3\beta n s_0 + 36 c_3 b^2 n r_{00} - 3c_3 r_{00} b^2 + 2c_2 r_{00} - 102 n r_{00} c_2 + 3564\beta^2 c_3^2 b^2 r_{00} c_2,\\
d_4 :&= 1728\beta^3 c_3^3 b^2 s_0 - 648\beta^2 c_3^3 b^4 r_{00} - 3264\beta^3 c_3^2 s_0 c_2 + 576\beta^2 c_3^2 r_{00} c_1 - 3006\beta^2 c_3 r_{00} c_2^2\\
&\quad + 1728\beta b^6 s_0 c_3^4 + 5184 c_3^3 b^4 r_{00} c_1 + 3564 c_3^3 b^6 r_{00} c_2 - 29376 c_3^3\beta b^4 s_0 c_2 - 12096 c_3^3\beta b^2 s_0 c_1\\
&\quad + 76608 c_3^2\beta b^2 s_0 c_2^2 + 21888 c_3^2\beta s_0 c_1 c_2 - 27054 c_3^2 b^4 r_{00} c_2^2 - 27216 c_3^2 b^2 r_{00} c_1 c_2 - 1944 c_3^2 r_{00} c_1^2\\
&\quad + 21708 c_3 r_{00} c_1 c_2^2 - 40640 c_3\beta s_0 c_2^3 + 41769 c_3 b^2 r_{00} c_2^3 - 14661 r_{00} c_2^4\\
&\quad + 3564\beta^2 c_3^2 b^2 r_{00} c_2 - 49248 c_3^2 b^2 c_1 c_2 n s_0 - 2880 c_3^2 c_1^2 n s_0 - 57456 c_3^2 b^4 c_2^2 n s_0 + 9072 c_3^3 b^4 c_1 n s_0\\
&\quad + 91440 c_3 b^2 c_2^3 n s_0 + 7344 c_3^3 b^6 c_2 n s_0 - 33085 c_2^4 n s_0 + 40632 c_3 c_1 c_2^2 n s_0 - 12 c_3^2 b^4\beta^2\\
&\quad + 36 c_3^2 b^4 n\beta^2 - 36 c_3 n c_1\beta^2 - 420 c_3 b^2 n c_2\beta^2 + 47 b^2 c_2 c_3\beta^2 - 24 c_3 c_1\beta^2 + 555 n c_2^2\beta^2 - 18 c_2^2\beta^2,\\
d_5 :&= -54\beta r_{00} c_3^2 - 42 b^4\beta^5 n r_{00} c_1^4 c_2^4 + 660 b^4\beta^5 r_{00} c_1^5 c_2^2 c_3 + 216 b^2\beta^6 n r_{00} c_1^4 c_2^4 - 56 b^2\beta^5 n r_{00} c_1^5 c_2^3\\
&\quad - 14 b^2\beta^5 r_{00} c_1^6 c_2 c_3 - 240 b^4\beta^5 r_{00} c_1^6 c_3^2 + 204 b^4\beta^5 r_{00} c_1^4 c_2^4 - 7 b^2\beta^5 r_{00} c_1^5 c_2^3,\\
d_6 :&= -1728 c_3^3\beta b^4 s_0 + 216 c_3^3 b^6 r_{00} - 1728 c_3^2 b^2 r_{00} c_1 - 3564 c_3^2 b^4 r_{00} c_2 + 9792 c_3^2 beta b^2 s_0 c_2\\
&\quad + 1344 c_3^2\beta s_0 c_1 + 9018 c_3 b^2 r_{00} c_2^2 + 3024 c_3 r_{00} c_1 c_2 - 8512 c_3\beta s_0 c_2^2 - 4641 r_{00} c_2^3 1728\beta b^6 s_0 c_3^4\\
&\quad + 5184 c_3^3 b^4 r_{00} c_1 + 3564 c_3^3 b^6 r_{00} c_2 - 29376 c_3^3\beta b^4 s_0 c_2 - 12096 c_3^3\beta b^2 s_0 c_1 + 76608 c_3^2\beta b^2 s_0 c_2^2\\
&\quad + 21888 c_3^2\beta s_0 c_1 c_2 - 27054 c_3^2 b^4 r_{00} c_2^2 - 27216 c_3^2 b^2 r_{00} c_1 c_2 - 1944 c_3^2 r_{00} c_1^2 + 21708 c_3 r_{00} c_1 c_2^2\\
&\quad - 40640 c_3\beta s_0 c_2^3 + 41769 c_3 b^2 r_{00} c_2^3 - 14661 r_{00} c_2^4 + 27 b^6 c_3^3 n s_0 - 459 b^4 c_2 c_3^2 n s_0 - 189 c_3^2 b^2 c_1\\
&\quad + 1197 b^2 c_2^2 c_3 n s_0 + 342 c_3 c_1 c_2 - 635 c_2^3 s_0 + 36 c_3 b^2 n r_{00} - 3b^2 c_3 r_{00} - 105 n c_2 + 2 c_2 r_{00}.
\end{aligned}$$

(3.3) implies that

$$d_6\alpha^4 + d_4\alpha^2 + d_2 = 0, \tag{3.4}$$

$$d_5\alpha^5 + d_0 = 0. \tag{3.5}$$

It follows from (3.5) that

$$r_{ij} = 0. \tag{3.6}$$

By putting (3.6) into (3.4) and simplifying the result, we have

$$\eta(x,y)s_0 = 0. \tag{3.7}$$

where

$$\begin{aligned}\eta(x,y) := &(1728\beta b^6 c_3^4 - 29376c_3^3\beta b^4 c_2 - 12096c_3^3\beta b^2 c_1 + 76608c_3^2\beta b^2 c_2^2 + 21888c_3^2\beta c_1 c_2 \\ &- 40640c_3\beta c_2^3 - 153b^2 c_2 c_3 - 21c_1 c_3 + 27b^4 c_3^2 + 133c_2^2 + 27b^6 c_3^3 n - 459b^4 c_2 c_3^2 n \\ &- 189c_3^2 b^2 c_1 + 1197b^2 c_2^2 c_3 n + 342c_3 c_1 c_2 - 635c_2^3)\alpha^4 + (51\beta^2 c_3 c_2 - 27\beta^2 c_3^2 b^2 \\ &- 12c_3^2 b^4\beta^2 + 36c_3^2 b^4 n\beta^2 - 36c_3 n c_1\beta^2 - 420c_3 b^2 n c_2\beta^2 + 47b^2 c_2 c_3\beta^2 - 24c_3 c_1\beta^2 \\ &+ 555nc_2^2\beta^2 - 18c_2^2\beta^2 + 1728\beta b^6 c_3^4 - 29376c_3^3\beta b^4 c_2 - 12096c_3^3\beta b^2 c_1 + 76c_3^2\beta b^2 c_2^2 \\ &+ 21888c_3^2\beta c_1 c_2 - 40640c_3\beta c_2^3 - 49248c_3^2 b^2 c_1 c_2 n - 2880c_3^2 c_1^2 n - 57456c_3^2 b^4 c_2^2 n \\ &+ 9072c_3^3 b^4 c_1 n + 91440c_3 b^2 c_2^3 n + 7344c_3^3 b^6 c_2 n - 33085c_2^4 n + 40632c_3 c_1 c_2^2 n)\alpha^2 \\ &+ 9\beta^4 c_3^2 - 48c_3\beta^4 n.\end{aligned}$$

By (3.7), it is obvious that $\eta = 0$ or $s_i = 0$. Let $\eta(x,y) = 0$. One can rewrite $\eta = 0$ as follows

$$\theta\alpha^4 + \gamma\alpha^2\beta^2 + \varepsilon\beta^4 = 0, \tag{3.8}$$

where $\theta = \theta(x,y)$, $\gamma = \gamma(x,y)$ and $\varepsilon = \varepsilon(x,y)$ are functions on TM. (3.8) implies that

$$\alpha^2 = \Big(\frac{-\gamma \pm \sqrt{\gamma^2 - 4\theta\delta}}{2\theta}\Big)\beta^2. \tag{3.9}$$

This contradicts with the positive-definiteness of α. Thus $\eta \neq 0$ and $s_i = 0$. □

Proof of Theorem 1.1: In [11] Najafi-Tayebi demonstrated that every weakly Landsberg (α,β)-metric with vanishing S-curvature on a manifold M of dimension $n \geq 3$ is a Berwald metric. By Theorem 1.1, every weakly Landsberg 5-th root metric on an $n(\geq 3)$-dimensional M is a Berwald metric. We consider the class 4-th (α,β)-metrics of dimension $n = 2$. We know that Every 2-dimensional Finsler manifold is C-reducible

$$C_{ijt} = \frac{1}{3}\Big\{h_{ij}I_t + h_{jt}I_i + h_{ti}I_j\Big\}. \tag{3.10}$$

Taking a horizontal derivation of (3.10) yields

$$L_{ijt} = \frac{1}{3}\Big\{h_{ij}J_t + h_{jt}J_i + h_{ti}J_j\Big\}. \tag{3.11}$$

By placing $\mathbf{J} = 0$ in (3.11), we have $\mathbf{L} = 0$. In other words, the Berwald curvature 2-dimensional Finsler manifold is written as follows

$$B^i{}_{jkt} = -\frac{2}{F}L_{jkt}l^i + \frac{2}{3}\Big\{E_{jk}h^i_t + E_{kt}h^i_j + E_{tj}h^i_k\Big\}. \tag{3.12}$$

By placing $\mathbf{L} = 0$ and $\mathbf{E} = 0$ in (3.12), we conclude that F is a Berwald metric. The proof is complete. □

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