Journa of Differential Geometry, Applications and Aspects

SOME SPECIAL TENSORS ON THE THREE-DIMENSIONAL WARPED PRODUCT MANIFOLDS

PARVANE ATASHPEYKAR

Department of Mathematics, Basic Sciences Faculty, University of Bonab, Bonab, Iran

Abstract.

In this article, we classify the tensors $\check{\mathcal{R}}$, $\check{\rho}$, and $\mathcal{R}[\rho]$ on particular class of the three-dimensional warped product manifolds. These tensors are used in the investigation of weakly-Einstein conditions on these manifolds. The concept of warped products is of particular importance in differential geometry and mathematical physics. This concept was first introduced by Bishop and ONiell to construct examples of Riemannian manifolds with negative curvature. In the following, warped product spaces have been extensively studied and used to construct new manifolds with interesting curvature properties. Also, in Lorentzian geometry, some well-known solutions to Einstein field equations, such as Schwarzschild and Friedmann-Robertson-Walker metrics, can be expressed in terms of warped products. Thus, Lorentzian warped products have been used to obtain more solutions to Einstein field equations. he warped products are of particular importance from a curvature point of view, Since in many cases they are related to the structure of the Codazzi tensors and sometimes locally conformally flat manifolds.

1. Introduction

Suppose (B, g_B) and (F, g_F) be two semi-Riemannian manifolds and $f : B \longrightarrow \mathbb{R}^+$ be a positive function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$, equipped with metric tensor $g = g_B \oplus f^2 g_F$. The function f is called the warping function of $M = B \times_f F$, B the base, and F the fiber. if $f = 1$, then $B \times_f F$ reduces to a semi-Riemannian product manifold. The concept of warped products is of particular importance in differential geometry and mathematical physics. This concept was first introduced by Bishop and ONiell to construct examples of Riemannian manifolds with negative curvature [1]. In the following, warped product spaces have been extensively studied and used to construct new manifolds with interesting curvature properties. Also, in Lorentzian geometry, some well-known solutions to Einstein field equations, such as Schwarzschild and Friedmann-Robertson-Walker metrics, can be expressed in terms of warped products. Thus, Lorentzian warped products have been used to obtain more solutions to Einstein field equations [2]. In this work, we want to compute the tensors $\check{\mathcal{R}}, \check{\rho}$ and $\mathcal{R}[\rho]$ on the three-dimensional Lorentzian warped product manifold $M = B \times_f F$ with metric

$$g = dt^2 + f^2 g_F, \tag{1.1}$$

where B is one-dimensional manifold and F is two-dimensional Lorentzian manifold. The tensors $\check{\mathcal{R}}, \check{\rho}$ and $\mathcal{R}[\rho]$ are symmetric $(0, 2)$-tensor fields which are defined as follows

$$\check{\mathcal{R}}_{ij} = \mathcal{R}_{iabc}\mathcal{R}_j^{abc}, \ \check{\rho}_{ij} = \rho_{ia}\rho_j^a, \ \mathcal{R}[\rho]_{ij} = \mathcal{R}_{iabj}\rho^{ab}. \tag{1.2}$$

These tensors are simplest tensors after Ricci tensor but they not seem to have received much attention in the literature (See for example, the discussion in [3]).

2010 *Mathematics Subject Classification.* 35B06, 34C14, 76M60.

Key words and phrases. Warped product manifold, Lorentzian three-metric, Curvature tensor, curvature identity.

2. Preliminaries

In this section, we first describe the basic concepts for the metric (1.1) to provide the necessary conditions for calculating the tensors $\check{\mathcal{R}}, \check{\rho}$ and $\mathcal{R}[\rho]$. See [4] for more information.

Lemma 2.1. *Let $M = B \times_f F$ be a Lorentzian warped product where B is one-dimensional manifold and F is two-dimensional Lorentzian manifold. If $U, V \in \mathfrak{X}(F)$, then*

(1) $\nabla_{\partial_t}\partial_t = 0$,
(2) $\nabla_{\partial_t}U = \nabla_U\partial_t = (\frac{f^{'}}{f})U$,
(3) $\mathrm{nor}(\nabla_U V) = II(U,V) = -g(U,V)(\frac{f^{'}}{f})\partial_t$,
(4) $\tan(\nabla_U V)$ *is the lift of* $\nabla^F_U V$ *on* F,

where $\nabla^F_U V$ is Levi-Civita connection on (F, g_F).

The next result provides the curvature of a warped product $M = B \times_f F$ in terms of its warping function f and the curvature tensor $\mathcal{R}^F$ of F.

Lemma 2.2. *Let $M = B \times_f F$ be a Lorentzian warped product where B is one-dimensional manifold and F is two-dimensional Lorentzian manifold. If $U, V, W \in \mathfrak{X}(F)$, then*

(1) $\mathcal{R}(U, \partial_t)\partial_t = \frac{H_f(\partial_t,\partial_t)}{f}U$,
(2) $\mathcal{R}(U, V)\partial_t = 0$,
(3) $\mathcal{R}(\partial_t, U)V = \frac{g(U,V)}{f}\nabla_{\partial_t}(\nabla f)$,
(4) $\mathcal{R}(U, V)W = \mathcal{R}^F(U, V)W - \frac{\|\nabla f\|^2}{f^2}(g(U,W)V - g(V,W)U)$.

From Lemma 2 we have the following.

Corollary 2.3. *Let $M = B \times_f F$ be a Lorentzian warped product where B is one-dimensional manifold and F is two-dimensional Lorentzian manifold. Then the Ricci tensor ρ of M satisfies*

(1) $\rho(\partial_t, \partial_t) = -\frac{2}{f}H_f(\partial_t, \partial_t)$,
(2) $\rho(U, \partial_t) = 0$,
(3) $\rho(U, V) = \rho^F(U, V) - g(U, V)(\frac{\triangle f}{f} + \frac{\|\nabla f\|^2}{f^2})$,

where ρ^F is the lift of the Ricci curvatures of F.

3. The tensor $\check{\mathcal{R}}, \check{\rho}$ and $\mathcal{R}[\rho]$ of warped product manifold

The warped products are of particular importance from a curvature point of view, Since in many cases they are related to the structure of the Codazzi tensors and sometimes locally conformally flat manifolds. We want to compute the tensors $\check{\mathcal{R}}, \check{\rho}$ and $\mathcal{R}[\rho]$ of the product warped $M = B \times_f F$ with equipped with metric tensor $g = dt^2 + f^2 g_F$, where F is two-dimensional Lorentzian manifold.

Theorem 3.1. *Suppose $M = B \times_f F$ be a Lorentzian warped product where B is one-dimensional manifold and $F$is two-dimensional Lorentzian manifold. If $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$, then the tensor $\check{\mathcal{R}}$ satisfies in the following conditions:*

(1) *The tensor $\check{\mathcal{R}}$ satisfies in the following conditions:*
  (1-1) $\check{\mathcal{R}}(X, Y) = \frac{4}{f^2}\|h_f\|^2 g(X, Y)$,
  (1-2) $\check{\mathcal{R}}(X, V) = 0$,
  (1-3) $\check{\mathcal{R}}(U, V) = \frac{1}{f^2}\check{\mathcal{R}}^F(U, V)$.
(2) *The tensor $\check{\rho}$ satisfies in the following conditions:*
  (2-1) $\check{\rho}(X, Y) = \frac{4}{f^2}\|h_f\|^2 g(X, Y)$,

(2-2) $\check{\rho}(X,V)=0$,
(2-3) $\check{\rho}(U,V)=\frac{1}{f^2}\check{\rho}^F(U,V)$.

(3) *The tensor* $\mathcal{R}[\rho]$ *satisfies in the following conditions:*
(3-1) $\mathcal{R}[\rho](X,Y)=\frac{1}{f}(\frac{\tau^F}{f^2}-2(\frac{\triangle f}{f}-\frac{\|\nabla f\|^2}{f^2}))H_f(X,Y)$,
(3-2) $\mathcal{R}[\rho](X,V)=0$,
(3-3) $\mathcal{R}[\rho](U,V)=-\frac{2}{f^2}\|h_f\|^2 g(U,V)+\frac{1}{f^2}\mathcal{R}^F[\rho](U,V)-(\frac{\triangle f}{f}-\frac{\|\nabla f\|^2}{f^2})\rho^F(U,V)$.

Proof. Consider a local basis $\{\partial_t, v_1, v_2\}$, where $\partial_t=\frac{\partial}{\partial_t}$ is the natural basis of B and $\{, v_1, v_2\}$ a local orthonormal basis of Ricci eigenvectors on F. Using Lemma 2, the components of the tensor $\check{\mathcal{R}}$ with respect to the local pseudo-orthonormal frame field $\{\partial_t, \frac{1}{f}v_1, \frac{1}{f}v_2\}$, on $M=B\times_f F$ are given by

$$\begin{aligned}
\check{\mathcal{R}}(X,Y) &= \mathcal{R}_{X\alpha\beta\gamma}\mathcal{R}_{Y\alpha\beta\gamma}g(\alpha,\alpha)g(\beta,\beta)g(\gamma,\gamma)\\
&= \mathcal{R}_{Xv_i\partial_t\gamma}\mathcal{R}_{Yv_i\partial_t\gamma}g(v_i,v_i)g(\partial_t,\partial_t)g(\gamma,\gamma)\\
&+ \mathcal{R}_{Xv_iv_j\gamma}\mathcal{R}_{Yv_iv_j\gamma}g(v_i,v_i)g(v_j,v_j)g(\gamma,\gamma)\\
&= g(-\frac{1}{f}H_f(X,\partial_t)v_i,\gamma)g(-\frac{1}{f}H_f(Y,\partial_t)v_i,\gamma)g(\gamma,\gamma)\\
&+ g(\frac{g(v_i,v_j)}{f}\nabla_X\nabla f,\gamma)g(\frac{g(v_i,v_j)}{f}\nabla_Y\nabla f,\gamma)g(\gamma,\gamma)\\
&= \frac{2}{f^2}H_f(X,\partial_t)H_f(Y,\partial_t)+\frac{2}{f^2}g(\nabla_X\nabla f,\gamma)g(\nabla_Y\nabla f,\gamma)\\
&= \frac{4}{f^2}\|h_f\|^2 g(X,Y).
\end{aligned}$$

$$\begin{aligned}
\check{\mathcal{R}}(U,V) &= \mathcal{R}_{U\alpha\beta\gamma}\mathcal{R}_{V\alpha\beta\gamma}g(\alpha,\alpha)g(\beta,\beta)g(\gamma,\gamma)\\
&= \mathcal{R}_{U\partial_t\partial_t\gamma}\mathcal{R}_{V\partial_t\partial_t\gamma}g(\partial_t,\partial_t)g(\partial_t,\partial_t)g(\gamma,\gamma)\\
&+ \mathcal{R}_{U\partial_tv_i\gamma}\mathcal{R}_{V\partial_tv_i\gamma}g(\partial_t,\partial_t)g(v_i,v_i)g(\gamma,\gamma)\\
&+ \mathcal{R}_{Uv_iv_j\gamma}\mathcal{R}_{Vv_iv_j\gamma}g(v_i,v_i)g(v_j,v_j)g(\gamma,\gamma)\\
&= g(\frac{1}{f}H_f(\partial_t,\partial_t)U,\gamma)g(\frac{1}{f}H_f(\partial_t,\partial_t)V,\gamma)g(\gamma,\gamma)\\
&+ g(\frac{-g(U,v_i)}{f}\nabla_{\partial_t}\nabla f,\gamma)g(\frac{-g(V,v_i)}{f}\nabla_{\partial_t}\nabla f,\gamma)g(\gamma,\gamma)\\
&+ g(\mathcal{R}^F(U,v_i)v_j-\frac{\|\nabla f\|^2}{f^2}[g(U,v_j)v_i-g(v_i,v_j)U],\gamma)\\
& g(\mathcal{R}^F(V,v_i)v_j-\frac{\|\nabla f\|^2}{f^2}[g(V,v_j)v_i-g(v_i,v_j)V],\gamma)
\end{aligned}$$

$$
\begin{aligned}
&= g(\mathcal{R}^F(U,v_i)v_j,v_k)g(\mathcal{R}^F(V,v_i)v_j,v_k)\\
&- \frac{\|\nabla f\|^2}{f^2}[g(V,v_j)g(v_i,v_k)-g(v_i,v_i)g(V,v_k)]g(\mathcal{R}^F(U,v_i)v_j,v_k)\\
&- \frac{\|\nabla f\|^2}{f^2}[g(U,v_j)g(v_i,v_k)-g(v_i,v_i)g(U,v_k)]g(\mathcal{R}^F(V,v_i)v_j,v_k)\\
&+ \frac{\|\nabla f\|^4}{f^4}[g(U,v_j)g(v_i,v_k)-g(v_i,v_i)g(U,v_k)][g(V,v_j)g(v_i,v_k)-g(v_i,v_j)g(V,v_k)]\\
&= \frac{1}{f^2}\check{\mathcal{R}}^F(U,V).
\end{aligned}
$$

The components of the tensor $\check{\rho}$ are obtained as follows:

$$
\begin{aligned}
\check{\rho}(X,Y) &= \rho(X,\alpha)\rho(Y,\beta)g(\alpha,\alpha)\\
&= \rho(X,\partial_t)\rho(Y,\partial_t)g(\partial_t,\partial_t)\\
&= \frac{4}{f^2}H_f(X,\partial_t)H_f(Y,\partial_t)\\
&= \frac{4}{f^2}\|h_f\|^2 g(X,Y).\\
\check{\rho}(U,V) &= \rho(U,\alpha)\rho(V,\beta)g(\alpha,\alpha)\\
&= \rho(U,v_i)\rho(V,v_i)g(v_i,v_i)\\
&= [\rho^F(U,v_i)-g(U,v_i)(\frac{\triangle f}{f}+\frac{\|\nabla f\|^2}{f^2})][\rho^F(V,v_i)-g(V,v_i)(\frac{\triangle f}{f}+\frac{\|\nabla f\|^2}{f^2})]g(v_i,v_i)\\
&= \frac{1}{f^2}\check{\rho}^F(U,V)-(\frac{\triangle f}{f}+\frac{\|\nabla f\|^2}{f^2})[g(U,v_i)\rho^F(V,v_i)+g(V,v_i)\rho^F(U,v_i)]\\
&\quad +(\frac{\triangle f}{f}+\frac{\|\nabla f\|^2}{f^2})[g(U,v_i)g(V,v_i)]g(v_i,v_i)\\
&= \frac{1}{f^2}\check{\rho}^F(U,V).
\end{aligned}
$$

With a similar process, we can obtain the components of the tensor $\mathcal{R}[\rho]$. □

References

[1] R. L. Bishop and B. ONeill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145**, 1969.

[2] J. K. Beem, P. Ehrlich and T. G. Powell, *Warped product manifolds in relativity in selected studies*, North-Holland, Armsterdarm, 1982.

[3] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematic und ihrer Grenzgebiete **93**, Springer, Berlin, New York, 1978.

[4] B. O'Neil, *Semi-Riemannian Geometry, with applications to relativity*, Academic Press, New York, 1983.