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PERTURBATED P-LAPLACIAN ON RIEMANNIAN MANIFOLDS

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ABSTRACT.

This paper deals with the nonlinear eigenvalue problem, for perturbated p-Laplacian operator, on a compact Riemannian manifold and determines a gradient estimate of eigenfunction associated with (first) eigenvalue of perturbated p-Laplacian operator. Using this estimate, we find a lower bound for this eigenvalue. In this paper we investigate the first (principal) nonlinear eigenvalue of the perturbated p-Laplacian on compact Riemannian manifolds and provide a lower bound through use of the diameter and the inscribed radius in terms of geometric quantities of manifold, and properties of disturbed term, when the Ricci curvature is non-negative. There are many results on the lower bound estimates for principal eigenvalues and eigenfunctions for domains in Euclidean space examined in multiple research papers. For a compact manifold with no boundary, for Laplace operator, i.e. $p = 2$, a sharp lower bound estimate on a compact Riemannian manifold with nonnegative Ricci curvature is known. Through a process of computation which involves Lagrange multipliers, it can be demonstrated.

1. INTRODUCTION

The main objective of this paper is to investigate the first (principal) nonlinear eigenvalue of the perturbated p-Laplacian on compact Riemannian manifolds and provide a lower bound through use of the diameter and the inscribed radius in terms of geometric quantities of manifold, and properties of disturbed term, when the Ricci curvature is non-negative. We denote by Δ_p the p-laplacian, i.e.

$$\Delta_p u = -div\left(|\nabla u|^{p-2}\nabla u\right),$$

and consider the following nonlinear eigenvalue problem:

$$\Delta_p u + F\left(x, u(x), \nabla u(x)\right) = -\lambda |u|^{p-2}u, \qquad u \not\equiv 0. \tag{1.1}$$

The examination of lower bound approximations for principal eigenvalues and eigenfunctions in Euclidean space domains has been a topic of interest in numerous research papers (e.g. [1],[2],[3],[4],[5], [6],[7]).

For a compact manifold with no boundary, we set $\lambda_{1,p}$ as the infimum of positive λ such that there is $u \not\equiv 0$ for which

$$\Delta_p u = -\lambda |u|^{p-2}u.$$

It has been determined in [8], that a compact Riemannian manifold with non-negative Ricci curvature has a sharp lower bound estimate for the Laplace operator, i.e. $p = 2$,

$$\lambda_{1,2} \geq \frac{\pi^2}{d^2}.$$

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Through a process of computation which involves Lagrange multipliers, it can be demonstrated that this is equivalent to

$$\lambda_{1,p} = \inf\left\{ \frac{\left(\int_M |\nabla u|^p\right)}{\left(\int_M |u|^p\right)} ; 0 \neq u \in W^{1,p}(M)\ ,\ \int_M |u|^{p-2}u = 0 \right\}. \tag{1.2}$$

Remark 1.1. Utilizing the same method, a similar expression is shown to be valid for 1.1.

Kawai and Nakauchi in [9], have shown that for a compact Riemannian manifold M without boundary and $p \geq 2$, if the inequality $Ric(M) \geq 0$ holds, then we have

$$\lambda_{1,p} \geq \frac{1}{p-1}\left(\frac{\pi}{4}\right)^p \frac{1}{d^p},$$

where, the diameter of M is denoted by d, while $Ric(M)$ stands for the Ricci curvature of M. The purpose of this paper is to present a gradient estimate for eigenfunction and by examining geometrical aspects and features of the perturbated term, a lower bounds for the principal eigenvalue of 1.1 can be calculated.

Proposition 1.2. *[gradient estimate for u] Let M be a compact Riemannian manifold with non-negative Ricci curvature and u be an eigenfunction in association with the eigenvalue λ of 1.1, then*

$$\frac{|\nabla u|^p}{\beta - |u|^p} \leq \frac{\lambda + F\left(x, u(x), \nabla u(x)\right)}{p-1},$$

where β having a value of $\mu \sup |u|$ and $\mu > 1$, is determined eventually.

Theorem 1.3. *Let M be a compact Riemannian manifold and $p \geq 2$. If $\lambda(= \lambda_{1,p})$ be eigenvalue of 1.1, then*

$$\lambda \geq \left[\frac{1}{d}\left(\frac{\pi\left((p-1)2^{p-1}\right)^{\frac{1}{p}}}{p\sin\left(\frac{\pi}{p}\right)} - C^{\frac{1}{p}}\beta\right) - C^{\frac{1}{p}}\right]^p,$$

under the following condition:
(a) the inequality $Ric(M) \geq 0$ holds,
(b) F is a Caratheodory function and $|F(x,\eta,\mu)| \leq C\left(1 + |\mu|\right)^p$,
(c) $F(.,0,0) = 0$.

The conventions we will use are as follows:
$(M, \langle .,. \rangle)$ represents a Riemannian manifold with nonnegative Ricci curvature, diameter d and dimension n. For fix $p > 1$, and a function $u : M \to \mathbb{R}$, $Hess$ will represent the Hessian as a $(2,0)$ or $(1,1)$ tensor, with a definition. We will use the convention

$$u_i := \nabla_{e_i} u \quad , \quad u_{ij} := \nabla_j \nabla_i u.$$

It is known that

$$\left(Hess\ u\right)\left(\nabla u, \nabla f\right) = \sum_{i,j} u_{ij} u^i f^j,$$

where u^i, f^j are elements of ∇u, ∇f respectively.
Our main results are determined by employing Li-Yau's gradient estimate method, in [10]. Using the linear operator $P(u)$, we define a continuous function $P(u)f$, for every function u in class C^1 and f in class C^2,

$$P(u)f = |\nabla u|^{p-2}\Delta f + (p-2)|\nabla u|^{p-4}\left(Hess\ f\right)\left(\nabla u, \nabla u\right),$$

and estimate $P(u)|\nabla u|^p$.
In this kind of expressions and computation, we have to deal with higher order derivatives. It is well known

that weak solutions of equations such as 1.1, belong to $W^{1,p}(M)\bigcap C^{1,\alpha}(M)$, for some $0<\alpha<1$. Under same conditions on F, such that F is a Caratheodory function and satisfy the following growth condition

$$|F(x,u,\nabla u)|\leq C\left(1+\|\nabla u\|\right)^p, \tag{1.3}$$

for some positive constant C and all $\mu\in\mathbb{R}^d-\{0\}$, for example see [4], [11], and 1.1 will never yield a nontrivial solution in class C^2, as stated by maximum principle. In fact, if u belongs to class C^2 everywhere, we can rephrase equation 1.1 accordingly:

$$|\nabla u|^{p-2}\Delta u+(p-2)|\nabla u|^{p-4}\left(Hess\ u\right)\left(\nabla u,\nabla u\right)+F(x,u,\nabla u)=-\lambda|u|^{p-2}u.$$

If $|\nabla u|=0$ at x_0, then by assumption on $F, u(x_0)=0$ and due to the compact nature of M, it follows that $\max u=\min u=0$, which is contrary to $u\not\equiv 0$.
For a good reference of this kind of results, one can refer to [11].

2. Linearization of p-Laplacian and lemmas

In this section, we shall prove some calculation lemmas.
In the beginning, we employ a naive approach to estimate the linearization of the p-Laplacian near u, i.e., for every function $u\in C^1(M)$ and $f\in C^2(M)$, let

$$\begin{aligned}
L(u)f\equiv\left.\frac{d}{dt}\right|_{t=0}\Delta_p(u+tf)&=div\left((p-2)|\nabla u|^{p-4}\left\langle\nabla u,\nabla f\right\rangle\nabla u+|\nabla u|^{p-2}\nabla f\right)\\
&=(p-2)\Delta_p u\frac{\langle\nabla u,\nabla f\rangle}{|\nabla u|^2}+(p-2)|\nabla u|^{p-2}\left\langle\nabla u,\nabla\frac{\langle\nabla u,\nabla f\rangle}{|\nabla u|^2}\right\rangle+\\
&\quad(p-2)|\nabla u|^{p-4}(Hess\ u)(\nabla u,\nabla f)+|\nabla u|^{p-2}\Delta f\\
&=|\nabla u|^{p-2}\Delta f+(p-2)|\nabla u|^{p-4}(Hess\ f)(\nabla u,\nabla u)+(p-2)\nabla_p u\frac{\langle\nabla u,\nabla f\rangle}{|\Delta u|^2}\\
&\quad+2(p-2)|\nabla u|^{p-4}(Hess\ u)\left(\nabla u,\nabla f-\frac{\nabla u}{|\nabla u|}\left\langle\frac{\nabla u}{|\nabla u|},\nabla f\right\rangle\right).
\end{aligned}$$

Now if u is an eigenfunction of the equation $\Delta_p u=-\lambda|u|^{p-2}u$, the pointwise definition of this operator only applies when the gradient of u is nonzero (and so u is locally smooth), so it can be easily demonstrated that it is strictly elliptic at these points.
For convenience, denote by $P(u)f$ the second order part of $L(u)$, which is

$$P(u)f\equiv|\nabla u|^{p-2}\Delta f+(p-2)|\nabla u|^{p-4}\left(Hess\ f\right)\left(\nabla u,\nabla u\right), \tag{2.1}$$

or equivalently

$$P(u)f\equiv\left[|\nabla u|^{p-2}\delta_{ij}+(p-2)|\nabla u|^{p-4}u_iu_j\right]f_{ij}. \tag{2.2}$$

The primary symbol of $P(u)$ is non-negative in all areas and strictly positive around points where ∇u is not null.
In regards to a local orthonormal frame field $\{e_1,\cdots,e_n\}$, we possess

$$P(u)f=\sum_{i,j}p_{ij}(u)f_{ij}, \tag{2.3}$$

where $p_{ij}(u)=|\nabla u|^{p-2}\delta_{ij}+(p-2)|\nabla u|^{p-4}u_iu_j$.
If u is of class $C^2(M)$, then $P(u)u=\Delta_p u$ and $L(u)u=(p-1)\Delta_p u$.
The aim of linearized p-Laplacian is to achieve a version of the recognized Bochner formula that can be applied to equation 1.1.

The following two lemmas are necessary. Identities $\left(|\nabla u|^2\right)_i = 2\sum_j u_j u_{ji}$ and $\nabla\left(|\nabla u|^2\right) = 2|\nabla u|\nabla|\nabla u|$, are straightforward.

Lemma 2.1. *If a function u is of class C^2, then*

$$\sum_{i,j} u_i u_j u_{ji} = \frac{1}{2}\left\langle \nabla u, \nabla|\nabla u|^2\right\rangle \leq |\nabla u|^2 \Big|\nabla|\nabla u|\Big|,$$
$$\sum_{i,j,k} u_i u_j u_{ki} u_{kj} = |\nabla u|^2 \Big|\nabla|\nabla u|\Big|^2.$$

Lemma 2.2. *In the case of a weak solution u to equation 1.1, the following identity hold:*

$$|\nabla u|^{p-2}\Delta u + F(x,u,\nabla u) = -\frac{p-2}{2}|\nabla u|^{p-4}\left\langle \nabla u, \nabla|\nabla u|^2\right\rangle - \lambda|u|^{p-2}u,$$

at every point, where $\nabla u \neq 0$.

Lemma 2.3. *When u is an eigenfunction of equation 1.1and $\nabla u \neq 0$:*

$$P(u)(|u|^p) = p(p-1)^2|u|^{p-2}|\nabla u|^p - \lambda p|u|^{2p-2} - pF(x,u,\nabla u)|u|^{p-2}u.$$

Proof. With an immediate calculation of 2.3, we can see that

$$\begin{aligned}(|u|^p)_{ij} &= p(p-2)|u|^{p-4}u^2 u_i u_j + p|u|^{p-2}u_i u_j + p|u|^{p-2}u u_{ij}\\ &= p(p-1)|u|^{p-2}u_i u_j + p|u|^{p-2}u u_{ij},\end{aligned}$$

so

$$\begin{aligned}P(u)\left(|u|^p\right) &= \sum_{i,j} p_{ij}(u)\left(|u|_{ij}\right)^p\\ &= p(p-1)|u|^{p-2}\left(|\nabla u|^p + (p-2)|\nabla u|^{p-4}\sum_{i,j} u_i^2 u_j^2\right) + p|u|^{p-2}u\left(P(u)u\right)\\ &= p(p-1)^2|u|^{p-2}|\nabla u|^p - \lambda p|u|^{2p-2} - pF(x,u,\nabla u)|u|^{p-2}u.\end{aligned}$$

□

Lemma 2.4. *For $u \in C^3(M)$,*

$$\begin{aligned}P(u)\left(|\nabla u|^p\right) &= p|\nabla u|^{2p-4}\sum_{i,k} u_{ki}^2 + p(p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_i u_j u_{ki} u_{kj}\\ &+ p|\nabla u|^{2p-4}\sum_k u_k(\Delta u)_k + p(p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_k u_i u_j u_{ikj}\\ &+ p|\nabla u|^{2p-4}\sum_{i,j} R_{ij}u_i u_j + p(p-2)|\nabla u|^{2p-6}\sum_{i,k,l} u_l u_k u_{li} u_{ki}\\ &+ p(p-2)^2|\nabla u|^{2p-8}\sum_{i,j,k,l} u_i u_j u_l u_k u_{lj} u_{ki}.\end{aligned}$$

Proof. First by the Ricci formula, we have $u_{kij} = u_{ijk} + \sum_l R_{jkil} u_l$, so

$$(|\nabla u|^p)_{ij} = \left(p|\nabla u|^{p-2} \sum_k u_k u_{ki} \right)_j = p|\nabla u|^{p-2} \sum_k u_{ki} u_{kj} + p|\nabla u|^{p-2} \sum_k u_k u_{kij} + p(p-2)|\nabla u|^{p-4} \sum_{k,l} u_l u_k u_{lj} u_{ki},$$

and

$$\begin{aligned} P(u)\left(|\nabla u|^p\right) &= \sum_{i,j} p_{ij}(u) \left(|\nabla u|^p\right)_{ij} = p|\nabla u|^{p-2} \sum_{i,j,k} p_{ij}(u) u_{ki} u_{kj} + \\ & p|\nabla u|^{p-2} \sum_{i,j,k} p_{ij}(u) u_k u_{ijk} + p(p-2)|\nabla u|^{p-4} \sum_{i,j,k,l} p_{ij}(u) u_l u_k u_{lj} u_{ki} \\ & + p|\nabla u|^{p-2} \sum_{i,j,k,l} p_{ij}(u) u_k R_{jkil} u_l = p(p-2)^2 |\nabla u|^{2p-8} \sum_{i,j,k,l} u_i u_j u_l u_k u_{lj} u_{ki} \\ & + p|\nabla u|^{2p-4} \sum_{i,k} (u_{ki})^2 + p(p-2)|\nabla u|^{2p-6} \sum_{i,j,k} u_i u_j u_{ki} u_{kj} \\ & + p|\nabla u|^{2p-4} \sum_{i,k} u_k u_{iik} + p(p-2)|\nabla u|^{2p-6} \sum_{i,j,k} u_k u_i u_j u_{ijk} \\ & + p|\nabla u|^{p-2} \sum_{i,j,k,l} p_{ij}(u) u_k R_{jkil} u_l + p(p-2)|\nabla u|^{2p-6} \sum_{i,k,l} u_l u_k u_{li} u_{ki}. \end{aligned}$$

However, it is established that $\sum_i u_{iik} = (\Delta u)_k$. Additionally, according to Bianchi's formula, term $\sum_{i,j,k,l} p_{ij}(u) u_k R_{jkil} u_l$, implies that $R_{ijkl} + R_{jkil} + R_{kijl} = 0$. Furthermore, considering the definition of $R_{kl} (:= \sum_i R_{ikil})$, it can be concluded that

$$\begin{aligned} \sum_{i,j,k,l} p_{ij} R_{jkil} u_k u_l &= |\nabla u|^{p-2} \sum_{i,k,l} R_{ikil} u_k u_l \\ & + (p-2)|\nabla u|^{p-4} \sum_l u_l \left(\sum_{i,j,k} u_i u_j u_k R_{jkil} \right) = |\nabla u|^{p-2} \sum_{k,l} R_{kl} u_k u_l. \end{aligned}$$

This equality finishes the proof. □

Lemma 2.5. *Suppose u is an eigenfunction of equation 1.1. At any point where $\nabla u \neq 0$, the following applies,*

$$\begin{aligned} |\nabla u|^p \sum_k u_k (\Delta u)_k + (p-2)|\nabla u|^{p-2} \sum_{i,j,k} u_i u_j u_k u_{ijk} &= -\lambda(p-1)|u|^{p-2} |\nabla u|^4 \\ & - (p-2)|\nabla u|^{p-2} \sum_{i,j,k} (u_k u_l u_{jk} u_{lj} + u_k u_j u_{lk} u_{lj}) + (p-2)F \sum_{i,k} u_i u_k u_{ik} \\ & + 2(p-2)|\nabla u|^{p-4} \sum_{i,j,k,l} u_i u_j u_k u_l u_{ik} u_{jl} - |\nabla u|^2 \sum_k u_k F_k \\ & + \lambda(p-2) u |u|^{p-2} \sum_{i,k} u_i u_k u_{ik}. \end{aligned}$$

Proof. It is easy to obtain

$$\Delta_p u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\sum_{j,l} u_j u_l u_{jl}.$$

By applying the $(.)_k$ on both sides of equation 1.1,

$$\begin{aligned}
&|\nabla u|^{p-2}(\Delta u)_k + (p-2)\Delta u|\nabla u|^{p-4}\sum_{i} u_i u_{ik} + \\
&\qquad (p-2)(p-4)|\nabla u|^{p-6}\sum_{i,j,l} u_i u_{ik} u_j u_l u_{jl} \\
&\qquad + (p-2)|\nabla u|^{p-4}\sum_{j,l}(u_{jk}u_l u_{jl} + u_j u_{lk} u_{jl} + u_j u_l u_{jlk}) \\
&\qquad + F_k = -\lambda(p-1)|u|^{p-2}u_k.
\end{aligned}$$

Given the operation of multiplying $|\nabla u|^2 u_k$ to both sides and applying summation on k, and considering the fact that u is an eigenfunction of equation 1.1, it can be deduced that

$$\begin{aligned}
&|\nabla u|^p\sum_k u_k(\Delta u)_k - (p-2)\left(\lambda u|u|^{p-2} + F + (p-2)|\nabla u|^{p-4}\sum_{j,l} u_j u_l u_{jl}\right)\times \\
&\quad \sum_{i,k} u_i u_k u_{ik} + (p-2)(p-4)|\nabla u|^{p-4}\sum_{i,j,k,l} u_i u_j u_k u_l u_{ik} u_{jl} + (p-2)|\nabla u|^{p-2}\times \\
&\quad \left(\sum_{j,k,l} u_k u_l u_{jk} u_{jl} + u_k u_j u_{lk} u_{jl} + u_k u_j u_l u_{jlk}\right) + |\nabla u|^2\sum_k u_k F_k \\
&\quad = -\lambda(p-1)|u|^{p-2}|\nabla u|^2\sum_k (u_k)^2.
\end{aligned}$$

This proves the lemma. □

Now we can obtain a formula of the Bochner-Weitzonbeak type for eigenfunctions of 1.1.

Proposition 2.6. *Let M be a Riemannian manifold of nonnegative Ricci curvature, and u be an eigenfunction of 1.1. Then when $\nabla u \neq 0$, the subsequent inequality is valid:*

$$\begin{aligned}
\frac{1}{p}P(u)\left(|\nabla u|^p\right) \geq{}& (p-1)^2|\nabla u|^{2p-4}A_u^2 + \lambda(p-2)|\nabla u|^{p-2}|u|^{p-2}uA_u \\
&+ (p-2)|\nabla u|^{p-2}A_u F - \lambda(p-1)|\nabla u|^p|u|^{p-2} - |\nabla u|^{p-2}\langle\nabla F, \nabla u\rangle \\
&+ (p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_i u_j (u_{ik} - u_{ki}) u_{jk},
\end{aligned}$$

where $A_u = \frac{|Hess\ u|(\nabla u, \nabla u)}{|\nabla u|^2}$.

Proof. Through the lemmas stated above, and with the calculation of the left side, we can conclude

$$\begin{aligned}
\frac{1}{p}P(u)\left(|\nabla u|^p\right) ={}& |\nabla u|^{p-2}\left[\langle\nabla\left(\Delta_p u\right), \nabla u\rangle - (p-2)A_u\Delta_p u\right] \\
&+ |\nabla u|^{2p-4}\left[|Hess\ u|^2 + Ric(M)(\nabla u, \nabla u)\right] \\
&+ (p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_i u_j (u_{ik} - u_{ki}) u_{jk}.
\end{aligned}$$

In the above computation, we note that, $Ric(M)(\nabla u, \nabla u) = \sum_{i,j} R_{ij} u_i u_j$, $|Hess\ u|^2 = \sum_{i,j}(u_{ij})^2$, and

$$-\lambda(p-1)|u|^{p-2}|\nabla u|^{4-p} - |\nabla u|^{2-p}\sum_k u_k F_k = |\nabla u|^{2-p}\left\langle \nabla\left(\Delta_p u\right), \nabla u\right\rangle.$$

Now in this identity, since $Ric(M) \geq 0, |Hess\ u|^2 \geq A_u^2$, and u is an eigenfunction of 1.1, we have

$$\begin{aligned}
&\frac{1}{p}P(u)\left(|\nabla u|^p\right) \geq |\nabla u|^{p-2}\left\langle \nabla\left(-\lambda|u|^{p-2}u - F(u)\right), \nabla u\right\rangle + (p-2)|\nabla u|^{p-2}A_u\times \\
&\left(\lambda|u|^{p-2}u + F\right) + (p-1)^2|\nabla u|^{2p-4}A_u^2 + (p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_i u_j (u_{ik} - u_{ki})u_{jk} \\
&= \lambda(p-2)|\nabla u|^{p-2}|u|^{p-2}uA_u + (p-2)|\nabla u|^{p-2}A_u F \\
&- \lambda|\nabla u|^{p-2}\left\langle \nabla\left(|u|^{p-2}u\right), \nabla u\right\rangle - |\nabla u|^{p-2}\left\langle \nabla F, \nabla u\right\rangle + (p-1)^2|\nabla u|^{2p-4}A_u^2 \\
&+ (p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_i u_j (u_{ik} - u_{ki})u_k \geq (p-1)^2|\nabla u|^{2p-4}A_u^2 \\
&+ \lambda(p-2)|\nabla u|^{p-2}|u|^{p-2}uA_u + (p-2)|\nabla u|^{p-2}A_u F - \lambda(p-1)|\nabla u|^p|u|^{p-2} \\
&- |\nabla u|^{p-2}\left\langle \nabla F, \nabla u\right\rangle + (p-2)|\nabla u|^{2p-6}\sum_{i,j,k} u_i u_j (u_{ik} - u_{ki})u_{jk}.
\end{aligned}$$

□

Remark 2.7. In order to improve the accuracy of our estimation, we can utilize a more sophisticated approach for calculating $|Hess\ u|^2$ in linear scenarios:

$$|Hess\ u|^2 \geq \frac{(\Delta u)^2}{n} + \frac{n}{n-1}\left(\frac{\Delta u}{n} - A_u\right)^2,$$

and get a better estimate.

3. Proof of main results

Our focus now shifts to establishing a gradient estimate for eigenfunctions. To begin, we will consider a weak solution u that belongs to the class $W^{1,p}(M)$ and fulfills equation 1.1. As stated $u \in C^{1,\alpha}$, for some $0 < \alpha < 1$.
The identity $\int_M u|u|^{p-2}d\nu = 0$, implies that u changes sign. Set $\beta = \mu(\sup_M u)^p$, where $\mu > 1$.

Proof of 1.2: We consider a continuous function $G = \frac{|\nabla u|^p}{\beta - |u|^p}$, $A = \beta - |u|^p$.
This function achieves it's maximum at a point $x_0 \in M$. We also assume that $\nabla u \neq 0$. From regularity of elliptic equations, G is smooth function around x_0.
The maximum principle implies that

$$\nabla G(x_0) = 0,$$
$$\left(P(u)G\right)(x_0) \leq 0.$$

To accurately represent the solution u, we take a local orthonormal frame field $\{e_1, \cdots, e_n\}$ centered around x_0, so we have $u_1 = |\nabla u|$, and $u_i = 0$, for $i \geq 2$. At x_0 , from $\nabla G = 0$ we have

$$\sum_j |\nabla u|^{p-2}u_j u_{ji} + |\nabla u|^p|u|^{p-2}u\frac{u_i}{A} = 0, \qquad for \quad i = 1, \cdots, n.$$

With above assumption on local orthonormal frame, we have

$$u_{1j} = 0 \qquad (for \quad j \geq 2), \quad u_1^{p-2}u_{11} = -u|u|^{p-2}G.$$

Employing lemmas in the previous section, we have

$$\begin{aligned} P(u)\left(|u|^{p}\right) &= p(p-1)^{2}|u|^{p-2}|\nabla u|^{p} - \lambda p|u|^{2p-2} - pF(x,u,\nabla u)|u|^{p-2}u \\ &= p(p-1)^{2}|u|^{p-2}u_1^{p} - \lambda p|u|^{2p-2} - pF|u|^{p-2}u \end{aligned}$$

and

$$\begin{aligned} P(u)\left(|\nabla u|^{p}\right) &= p|\nabla u|^{2p-4}\sum_{i,k}u_{ki}^{2} + p(p-2)|\nabla u|^{2p-6}\sum_{i,j,k}u_iu_ju_{ki}u_{kj} \\ &+ p|\nabla u|^{2p-4}\sum_{k}u_k(\Delta u)_k + p(p-2)|\nabla u|^{2p-6}\sum_{i,j,k}u_ku_iu_ju_{ikj} \\ &+ p|\nabla u|^{2p-4}\sum_{i,j}R_{ij}u_iu_j + p(p-2)|\nabla u|^{2p-6}\sum_{i,k,l}u_lu_ku_{li}u_{ki} \\ &+ p(p-2)^{2}|\nabla u|^{2p-8}\sum_{i,j,k,l}u_iu_ju_lu_ku_{lj}u_{ki} = pu_1^{2p-4}\left(\sum_{i,k}u_{ki}^{2} + (p-2)\sum_{k}u_{k1}^{2}\right) \\ &+ pu_1^{2p-3}(\Delta u)_1 + p(p-2)u_1^{2p-3}u_{111} + pu_1^{2p-2}R_{11} + p(p-2)u_1^{2p-4}u_{11}^{2} \\ &+ p(p-2)^{2}u_1^{2p-4}u_{11}^{2} = pu_1^{2p-4}\left(\sum_{i,k}u_{ki}^{2} + (p-2)\sum_{k}u_{k1}^{2}\right) \\ &+ pu_1^{2p-3}\left((\Delta u)_1 + (p-2)u_{111}\right) + pu_1^{2p-2}R_{11} + p(p-2)(p-1)u_1^{2p-4}u_{11}^{2} \end{aligned}$$

that is $Ric(M) \geq 0$, and $\sum_{i,k}u_{ki}^{2} \geq \sum_{k}u_{k1}^{2} > u_{11}^{2}$,

$$P(u)\left(|\nabla u|^{p}\right) > p(p-1)^{2}u_1^{2p-4}u_{11}^{2} + pu_1^{2p-3}\left((\Delta u)_1 + (p-2)u_{111}\right).$$

From lemma 2.5,

$$\begin{aligned} u_1^{p+1}(\Delta u)_1 + (p-2)u_1^{p+1}u_{111} &= -\lambda(p-1)u_1^{4}|u|^{p-2} - 2(p-2)u_1^{p}u_{11}^{2} + (p-2)Fu_1^{2}u_{11} \\ &+ 2(p-2)u_1^{p}u_{11}^{2} - u_1^{p-1}F_1 + \lambda(p-2)u|u|^{p-2}u_1^{2}u_{11}, \end{aligned}$$

or

$$\begin{aligned} u_1^{p+1}\left((\Delta u)_1 + (p-2)u_{111}\right) = (p-2)Fu_{11}^{2}u_{11} - u_1^{p-1}F_1 - \lambda(p-1)u_1^{4}|u|^{p-2} \\ + \lambda(p-2)uu_1^{2}u_{11}|u|^{p-2}. \end{aligned}$$

Recall the identity $|\nabla u|^{p} = AG$. Applying $P(u)$ to both sides $\nabla G(x_0) = 0$ and $P(u)G(x_0) \leq 0$,

$$\begin{aligned} P(u)\left(|\nabla u|^{p}\right) = P(u)(AG) &= \sum_{i,j}p_{ij}\left(A_{ij}G + A_iG_j + A_jG_i + AG_{ij}\right) \\ &= |\nabla u|^{p-2}\sum_{i}\left(A_{ii}G + 2A_iG_i + AG_{ii}\right) \\ &+ (p-2)|\nabla u|^{p-4}\sum_{i,j}\left(A_{ij}G + A_iG_j + A_jG_i + AG_{ij}\right). \end{aligned}$$

Then at x_0,

$$P(u)\left(|\nabla u|^p\right)(x_0) = |\nabla u|^{p-2}\sum_i \left(A_{ii}G + AG_{ii}\right) + (p-2)|\nabla u|^{p-4}u_1^2\left(A_{11}G + AG_{11}\right)$$
$$= u_1^{p-2}\left[\sum_i \left(A_{ii}G + AG_{ii}\right) + (p-2)A_{11}G + (p-2)AG_{11}\right].$$

But in x_0, $P(u)G \leq 0$ so $\sum_{i,j} p_{ij}G_{ij} \leq 0$ and then

$$(p-2)G_{11} + \sum_i G_{ii} \leq 0.$$

Hence

$$P(u)\left(|\nabla u|^p\right)(x_0) \leq u_1^{p-2}\left[(p-2)A_{11} + \sum_i A_{ii}\right]G.$$

Also $A = \beta - |u|^p$ implies that

$$A_{ii} = -p\left[(p-2)|u|^{p-4}u^2u_i^2 + |u|^{p-2}u_i^2 + |u|^{p-2}uu_{ii}\right].$$

For this point x_0,
if $i = 1$, $A_{11} = -p(p-1)|u|^{p-2}u_1^2 - p|u|^{p-2}uu_{11}$
and for $i > 1$, $A_{ii} = -p|u|^{p-2}uu_{ii}$.
Therefore,

$$P(u)\left(|\nabla u|^p\right)(x_0) \leq -pG|u|^{p-2}u_1^{p-2}[(p-1)u_1^2 + |u|^{p-2}uu_{11} + \sum_i uu_{ii} + (p-2)uu_{11}$$
$$+ (p-1)(p-2)u_1^2] = -G\left[p(p-1)^2|u|^{p-2}u_1^p - \lambda p|u|^{2p-2} - pFu|u|^{p-2}\right],$$

and then

$$P(u)\left(|\nabla u|^p\right)(x_0) \leq -GP(u)\left(|u|^p\right)(x_0).$$

Combining the above inequalities, at x_0,

$$-GP\left(|u|^p\right) \geq P\left(|\nabla u|^p\right),$$

so

$$-G\left(p(p-1)^2|u|^{p-2}u_1^p - \lambda p|u|^{2p-2} - pF|u|^{p-2}u\right) > p(p-1)^2u_1^{2p-4}u_{11}^2$$
$$+ pu_1^{2p-3}\left((\Delta u)_1 + (p-2)u_{111}\right) = p(p-1)^2u_1^{2p-4}u_{11}^2$$
$$+ pu_1^{p-4}\left((p-2)Fu_1^2u_{11} - u_1^{p-1}F_1 - \lambda(p-1)u_1^4|u|^{p-2} + \lambda(p-2)uu_1^2u_{11}|u|^{p-2}\right).$$

Finally

$$u_1^{2p-5}F_1 + \lambda\beta(p-1)G|u|^{p-2} + (p-1)u|u|^{p-2}FG > \beta G^2(p-1)^2|u|^{p-2},$$

or

$$\frac{u_1^{2p-5}}{(p-1)G\beta|u|^{p-2}}\|F_1\|_\infty + \lambda + \frac{u}{\beta}F > G(p-1).$$

Now as for boundedness of $\|F_1\|_\infty, \|F\|_\infty$, as well as u_1, $|u|$ at the point x_0, while μ is sufficiently near to 1, we are able to pass from the first term and coefficient of F and get $\lambda + F \geq G(p-1)$. □

Proof of 1.3. First, recall a well-known inequality in which for any $n > 1$:

$$|a+b|^n \leq 2^{n-1}\left(|a|^n + |b|^n\right).$$

Let $x_1, x_2 \in M$, such that $0 < u(x_1) = \sup u$, $u(x_2) = 0$, and consider a minimal normal geodesic γ, whit $\gamma(0) = x_2$ to $\gamma(1) = x_1$.
Due to the fact that $dist(x_2, x_1) \leq d$, and with the considering circumstance 1.3 on F, we have

$$\lambda + C\left(1+|\nabla u|\right)^p \geq (p-1)G$$

$$\sqrt[p]{\frac{\lambda}{(p-1)2^{p-1}}} + \sqrt[p]{\frac{C}{(p-1)2^{p-1}}}(1+|\nabla u|) \geq \frac{|\nabla u|}{(\beta - |u|^p)^{\frac{1}{p}}}.$$

Integrating along γ from x_2 to x_1 and then changing the variable, we obtain that

$$d\frac{\lambda^{\frac{1}{p}} + C^{\frac{1}{p}}}{((p-1)2^{p-1})^{\frac{1}{p}}} + \frac{C^{\frac{1}{p}}}{((p-1)2^{p-1})^{\frac{1}{p}}}\int_{x_2}^{x_1}|\nabla u| \geq \int_{x_2}^{x_1}\frac{|\nabla u|}{(\beta - |u|^p)^{\frac{1}{p}}}$$

or after simplification

$$d\frac{\lambda^{\frac{1}{p}} + C^{\frac{1}{p}}}{((p-1)2^{p-1})^{\frac{1}{p}}} + \frac{C^{\frac{1}{p}}}{((p-1)2^{p-1})^{\frac{1}{p}}}\int_{0}^{\sup u} du \geq \int_{0}^{\sup u}\frac{du}{(\beta - |u|^p)^{\frac{1}{p}}}$$

$$= \int_0^1 \frac{du}{(1-|u|^p)^{\frac{1}{p}}} = \frac{\pi}{p\sin(\frac{\pi}{p})}.$$

Therefore,

$$\lambda \geq \left[\frac{1}{d}\left(\frac{\pi\left((p-1)2^{p-1}\right)^{\frac{1}{p}}}{p\sin\left(\frac{\pi}{p}\right)} - C^{\frac{1}{p}}\beta\right) - C^{\frac{1}{p}}\right]^p.$$

□

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