NEW EXACT NON-REDUCIBLE SOLUTIONS FOR GENERALIZED ZAKHAROV-KUZNETSOV EQUATION

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Abstract.

In this paper, we obtain new non-reducible exact solutions of generalized Zakharov-Kuznetsov equation by method of partially invariant solutions (PISs). PISs method is the generalization of the similarity reduction method. We focus on the case of PISs that have defect structure 1 and they obtained from three-dimensional subalgebras. For this purpose, we calculate the optimal system of 1, 2 and 3 dimensional subalgebras of the symmetry algebra for the equation. Also, it will be shown that these solutions are different from the group invariant solutions computed by the method of Lie symmetry and their non-reducibility is proven.

1. Introduction

Nonlinear evolution equations represent a powerful mathematical framework for modeling dynamic processes that deviate significantly from linear behavior. These equations have remarkable applications in fields as diverse as fluid dynamics, quantum mechanics, pattern formation, and information theory. It is interesting to note that solving equations can be quite daunting at times. Sophus Lie introduced the classical Lie method or the Lie symmetry method in the 19th century [1]. This powerful method can reduce the order of ordinary differential equations [2, 3] and convert PDEs to ODEs in certain cases [4–6]. Furthermore, finding symmetry and a specific solution can lead to a wide range of solutions. The Lie symmetry group theory has been extensively used to analyze and solve PDE systems [7–9]. However, the Lie classical symmetry method has its limitations and cannot find all similarity reductions for PDE equations. Therefore, the development of new generalizations of this method has been motivated [10, 11]. With the new generalizations, the Lie symmetry method can be even more effective in solving complex problems. The method of partial invariant solutions (PISs) is applied to reduce PDEs. Like the similarity reduction method, this method is algorithmic and based on classification of subgroups of symmetry group. PISs, introduced by Ovsiannikov in 1992 [12], represent an extension of invariant solutions. The construction algorithm for PISs is identical to that of invariant solutions. By leveraging PISs, we can unlock a wide range of possibilities that were not available before. When we work with low-dimensional groups, it is important to note that the process of obtaining invariant solutions through the use of the partial invariant solutions method is generally more facile compared to the Similarity Reduction (SR) method. The Lie symmetry method is also commonly used to solve these equations, but it is often more challenging. Therefore, the PISs method presents an attractive alternative for researchers seeking a more straightforward method of obtaining invariant solutions.

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Through the constructing PISs, one important concept appears which is defect structure, denoted by σ. It is a quantity determined by dimension of orbits.

In this paper, we will use the PISs method to make new exact solutions of the generalized Zakharov-Kuznetsov equation (gZK). The Zakharov-Kuznetsov (ZK) equation is given as follow,

$$u_t + \alpha u u_x + \beta(u_{xx} + u_{yy})_x = 0. \tag{1.1}$$

This equation is a well-known 2-dimensional generalization of the KdV equation that describes the behavior of weakly nonlinear ion-acoustic waves in plasma consisting of cold ions and hot isothermal electrons in the presence of a uniform magnetic field [13]. The equation is called the generalized Zakharov-Kuznetsov equation and is given by

$$u_t + \alpha u^n u_x + \beta(u_{xx} + u_{yy})_x = 0, \tag{1.2}$$

where u is a smooth function with respect to (t, x, y) and α, β and n are arbitrary constants [14]. The initial term in the equation represents the evolution term, while the second term denotes the nonlinear term. The third and fourth terms, when taken together in parentheses, represent the dispersion terms. Solitons are the outcome of a subtle balance between dispersion and nonlinearity. The exponent n, which constitutes the power law nonlinearity parameter, is a positive real number. It seems that the extended tanh method was utilized in [15] to derive periodic solutions and solitons for (1.2), which may prove useful in describing wave characteristics in the field of plasma physics. On the other hand, complex solutions for (1.2) were obtained using the Cole-Hopf transformation and the first integral technique in [16]. We can change the variables and rewrite the gZK equation as a system of PDEs that follows:

$$\begin{aligned} &v - u_{xx} - u_{yy} = 0, \\ &u_t + \alpha u^n u_x + \beta v_x = 0. \end{aligned} \tag{1.3}$$

This system has a 4-dimensional Lie symmetry algebra $\mathfrak{g}$ generated by following infinitesimal generators [17].

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = n(3t\partial_t + x\partial_x + y\partial_y) - 2u\partial_u. \tag{1.4}$$

Partial invariant non-reducible solutions for system 1.3 are not invariant to lower-dimension subalgebras. Finding these PISs can help obtain new solutions for the generalized Zakharov-Kuznetsov equation. This paper is organized as follows. In section 2, the algorithm of finding PIS is presented. In section 3, we classify the three-dimensional Lie symmetry subalgebra of $\mathfrak{g}$. In section 4, we calculate some of non-reducible PISs for system 1.3.

2. Computing the PISs for a PDEs system

In this section, after stating the basics and important definitions, we will discuss the method of finding the partial invariant solutions for a system of partial differential equations [12, 18]. Consider a system of partial differential equations (PDEs) denoted by Δ, where the system is of nth order and has p independent variables $(x = (x^i) \in X, i = 1, ..., p)$ and q dependent variables $(u(u^j) \in U, j = 1, ..., q)$

$$\Delta = \Delta_\mu(x, u^{(n)}) = 0, \qquad \mu = 1, ..., r. \tag{2.1}$$

Assume that G serves as a local symmetry group for the aforementioned system. A local symmetry group is a set of transformations that preserve certain system properties locally, i.e., within a small neighborhood around each point. The orbit space of Γ_h can be defined as the set of r-dimensional orbits of G on $X \times U$ which intersect the graph Γ_h. This orbit space provides a natural parametrization of the solutions of the system (2.1) with graph Γ_h as

$$G\Gamma_h = \{g.(x, u) \mid (x, u) \in \Gamma_h, g \in G\},$$

The solution's defect structure for group G is the union of the orbits of its Γ_h-elements, where $u = h(x)$. So we have

$$\sigma = \dim(G\Gamma_h) - \dim(\Gamma_h) = \dim(G\Gamma_h) - p,$$

and calculated by the matrix of generators characteristics. Also we have

$$\min\{r, q\} \geq \sigma \geq 0.$$

If $\sigma = 0$, then $u = h(x)$ is an invariant solution. If $\min\{r, q\} > \sigma > 0$, then $u = h(x)$ is definitely a partially invariant solution. To calculate the PISs, it is imperative that we first categorize the symmetry group into conjugacy classes. To obtain PISs with a defect structure σ, we must select subgroups H that have the following effect: If the dimension of the H orbits on the space $X \times U$ is r, then the dimension of the orbits is $r - \sigma$ on the space X. This concept was originally mentioned in [19, 20], and it is essential to select H as a subgroup with this effect, and $\mathfrak{h}$ as its Lie algebra with infinitesimal generators $\{v_1, ..., v_s\}$. A complete set of functionally independent invariants can be obtained in the form of:

$$\{I_j(x, u), \xi_i(x)\}, \tag{2.2}$$

where $j = 1, ..., q - \sigma$ and $i = 1, ..., p + \sigma - s$. If $u = h(x)$ is a function, then we can express the space $H\Gamma_h$ according to the invariant (2.2). Thus

$$h_j(\xi_i(x)) = I_j(x, u), \tag{2.3}$$

where h_j are optional functions. Now by utilizing the implicit function theorem, we can express

$$u^{i_\alpha} = U^{i_\alpha}(x, u^{j\beta}, h_j(\xi_i(x))), \tag{2.4}$$

where $\beta = 1, ..., \sigma$ and $\alpha = 1, ..., q - \sigma$. It is noteworthy that the residuary dependent variables are solely dependent on the initial independent variables

$$u^{j_\beta} = U^{j_\beta}(x_1, ..., x_p), \qquad \beta = 1, ..., \sigma. \tag{2.5}$$

We need to find the derivatives of the functions $u^1, ..., u^q$ concerning recent variables that are received from equations (2.4) and (2.5). Once we substitute these amounts into the initial system, we get a revived system comprising the $q - \sigma$ functions h_j and invariants ξ_i. In general, the obtained equations are inconsistent, so we need to calculate the compatibility conditions. These constraints provide us with a system of PDEs, represented by Δ/H. Additionally, a PDEs system is obtained from (2.5), indicated by Δ'. Now, first, we should solve the system Δ/H, and for the individual solution of this system, solve the system Δ'. Then replace the solutions into equations (2.4) and (2.5) to receive the partially invariant solutions.

3. Classifying of symmetry algebras for (1.3)

We obtain the optimal system for symmetry subalgebras of equations that appear in (1.3) in this section. For this purpose, we categorize into conjugacy classes the subgroups of the symmetry group of (1.3), that is equivalent to classify the subalgebras of 1.3. In this paper, the classification of three-dimensional subalgebras is necessary to estimate the PISs with defect $\sigma = 1$, and where the decreased system Δ/H is a system of ODE. Given that $p = 3$, $\sigma = 1$, and $p + \sigma - s = 1$, we can conclude that $s = 3$, thus leading to the consideration of three-dimensional subgroups. Infinitesimal symmetries can be combined linearly to create an unlimited number of one-dimensional subalgebras for a given system. To determine which subgroups give different types of solutions, it's essential to find invariant solutions that aren't linked by transformation in the total group of symmetry. This approach directed to the idea of a subalgebras optimal system. For subalgebras in dimensional one, the classification issue is equivalent to classifying the orbits of the adjoint representation [21]. A solution to this problem is to take a general element in the Lie algebra and simplify it by imposing various adjoint transformations on it. Optimal sets of subalgebras can be obtained by selecting only one representative from each class of equivalent subalgebras [12].

The optimal system of (1.3) was calculated in [22] as follows:

1) One-dimensional optimal system:

$$X_1+\varepsilon X_2+\varepsilon X_3, \qquad X_2, \qquad X_3+aX_2, \qquad X_4, \qquad X_1, \tag{3.1}$$

where a is constant and $\varepsilon=0,\pm1$.

2)To classify subalgebras in dimension two, we should choose two generators. One generator is chosen from the inventory of optimal systems in dimension one , while the other generator is optional. Let $\mathfrak{h}=\operatorname{span}\{X,Y\}$ be a subalgebra of $\mathfrak{g}$ in dimension two, where X is a subalgebra in dimension one chosen from (3.1), and Y is an optional vector $Y=b_1X_1+\cdots+b_4X_4$. We need to facilitate $\mathfrak{h}$ by applying various adjoint transformations. By this method, two-dimensional optimal system is obtained as follows:

$$\begin{array}{llll}\langle X_1,X_2\rangle, & \langle X_1,X_3+aX_2\rangle, & \langle X_2,X_3\rangle, & \langle X_3,X_1+\varepsilon X_2\rangle,\\ \langle X_1,X_4\rangle, & \langle X_4,X_3+aX_2\rangle, & \langle X_4,X_2\rangle, & \langle X_1+\varepsilon X_3,X_2+aX_3\rangle.\end{array} \tag{3.2}$$

3) By the same method, three-dimensional optimal system is:

$$\langle X_2,X_3,X_4\rangle, \qquad \langle X_1,X_2,X_3\rangle, \qquad \langle X_1,X_4,X_3+aX_2\rangle, \qquad \langle X_1,X_2,X_4\rangle. \tag{3.3}$$

4. Non-reducible PISs for system 1.3

In this section we will perform the non-reducible partial invariant solutions. For a partial invariant solution $u=h(x)$ that is reducible, there exist subgroups of G that are not invariant with respect to u. However, we can discover a subgroup H', where is a subset of H, such that u is an H'-invariant solution and

$$s-\sigma=\dim(H)-\sigma\leqslant\dim(H'\Gamma_h).$$

By utilizing the method of similarity reduction, one can easily obtain reducible PISs from a reduced system that involves independent variables. This approach is not only efficient but also proves to be an effective way of obtaining desired outcomes from decreased system concerning

$$p+\sigma-s\geqslant p-\dim(H'\Gamma_h),$$

independent variables. Obtaining them through this method is easier than using the PISs method [12]. Assume the Lie subalgebra $\langle X_2,X_3,X_4\rangle$. The collection of functionally independent invariants for this subalgebra is a collection of functions I with the effect that $X_2(I)=X_3(I)=X_4(I)=0$. To obtain this set of functions, we can calculate them by solving for I using the above property. Once we have this set of functions, we will have the set of functionally independent invariants for this Lie subalgebra:

$$\{t,ut^{\frac{2}{3n}},v\}. \tag{4.1}$$

The corresponding equation to relations (2.3) is

$$ut^{\frac{2}{3n}}=h(v), \tag{4.2}$$

and we have the following expression for the solutions corresponding to equations (2.4) and (2.5)

$$u=t^{-\frac{2}{3n}}h(v), \qquad v=u^{-1}g(t). \tag{4.3}$$

We can now calculate the derivatives of functions u and v using the equations stated in (4.3). Substituting into the system (1.3) we obtain

$$\begin{aligned}&3n(-v+D(h(v))v_tt^{\frac{-2}{3n}})-2h(v)t^{\frac{-2-3n}{3n}}=0,\\ &3nt^{-\frac{2}{3n}}(D(h(v))v_y+\beta D^{(3)}(h(v))v_tv_x^2+2\beta D^{(2)}(h(v))v_xv_{tx}+\\ &\qquad\beta D^{(2)}(h(v))v_tv_{xx}+\beta D(h(v))v_{txx})+3n(v\alpha h^n(v)t^{\frac{-2}{3}}+\beta v_{tt})-\\ &\qquad 2\beta t^{\frac{-2-3n}{3n}}(D(h(v))v_{xx}+D^{(2)}(h(v))v_x^2)=0.\end{aligned} \tag{4.4}$$

By using the chain rule the consistency conditions obtained from system (4.4)

$$\begin{aligned} &-v_2^2 h + g^2 + 2vgg' = 0, \\ &2vgg' + vh' - v^2 {g'}^2 - v^2 g^2 = 0, \\ &v^2 h g'' - v g' g + v^2 g g'^2 - v^2 h g = 0. \end{aligned} \tag{4.5}$$

Solving this system of ODEs and by the use of the relations (4.3) we obtain these solutions:

$$u(t,x,y) = \sqrt[n]{\frac{c_1(n+1)(n+2)}{2\alpha}} \cosh^{-\frac{2}{n}} \left(\frac{n\sqrt{c}_1(c_2 + x - c_1 t - (c_3 - c_1)y)}{2\sqrt{\beta(1+(c_3-c_1)^2)}}\right), \tag{4.6}$$

where c_1, c_2, c_3 are constants. This solution is not similar to the invariant solutions which obtained in [23] by the Lie symmetry method. Also, these solutions are not equivalent to any of the previous solutions. So we deduce that PIS in (4.6) are new and non-reducible PISs. In (3.3), further subalgebras can be used to calculate non-reducible PISs in a similar method. These non-reducible partial invariant solutions are presented in Table I. It should be noted that c_1, c_2, c_3, c_4 in this table are arbitrary constants.

Table I

Subalgebras	PISs
$\langle X_1, X_2, X_3 \rangle$	$u(t,x,y) = \sqrt{\frac{-2c_1^2\alpha^2}{\alpha c_1^2(\beta c_1 - c_2)}} csc\big(c_4\sqrt{2c_1^2\alpha^2} \pm \frac{\alpha y - \beta x}{\alpha+1}\sqrt{\frac{-2c_1^2\alpha^2}{2(\alpha\beta c_3^3 - \alpha c_2 c_1^2 + \beta^2 c_1^2 - 2\beta c_2 c_1 + c_2^2)}}\big)$
$\langle X_1, X_4, X_3 + aX_2 \rangle$	$u(t,x,y) = \sqrt{\frac{-\alpha}{a(c_1\beta - c_2)}} tanh\big(c_1(\frac{t-\beta y}{\beta + na}\alpha + 1)\sqrt{\frac{n}{2(c_2^2 + \beta^2 c_1^2 - 2c_2\beta c_1 + \beta\alpha c_1^3 - c_1^2 c_2 \alpha)}}\big)$
$\langle X_1, X_2, X_4 \rangle$	$u(t,x,y) = \sqrt{\frac{2\beta}{n}} sec\big(\sqrt{2}\alpha\beta^{\frac{3}{2}}\big(\pm\frac{\alpha\beta t - \alpha x + y}{\alpha\beta\sqrt{2(\beta\alpha^2-1)}} + c_1\big)\big)$

Conclusion

In this paper, we have obtained new exact solutions of the generalized Zakharov-Kuznetsov equation by using the partially invariant solutions method. Partial invariance is a concept that plays a crucial role in many fields. While some partially invariant solutions behave consistently across all subalgebras, others exhibit non-reducible characteristics. These non-reducible PISs are particularly fascinating, as they possess unique properties and can offer valuable insights into complex problems. Non-reducible PISs are partially invariant solutions that cannot be reduced to lower-dimensional subalgebras. These non-reducible PISs provide new solutions for the system of partial differential equations. The solutions obtained through this method cannot be constructed using the similarity solution method.

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